MAT 102: Ordinary Differential Equations

Topic 1: Introduction to Differential Equations

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1 Introduction to Differential Equations

1.1 Definition of a Differential Equation

A **differential equation (DE)** is an equation that contains one or more derivatives of an unknown function.

$$F\big(x, y, y', y'', \dots, y^{(n)}\big) = 0$$

The **unknown** is a function (not a number), and the equation relates that function to its own derivatives.

Examples:

$$\frac{dy}{dx} = 3x^2 \qquad \text{(1st order ODE)}$$

$$\frac{d^2y}{dx^2} + 4y = 0 \qquad \text{(2nd order ODE)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace PDE)}$$

1.2 Some Basic Mathematical Models

Differential equations arise naturally from physical, biological, and social phenomena. Some important models include:

1.2.1 Population Growth (Malthusian Model)

$$\frac{dP}{dt} = kP$$

- (P(t)): population at time (t)
- (k): growth rate constant

Explanation:
The rate of population growth is proportional to the current population.
- More people → faster growth
- Fewer people → slower growth

Behavior:
- If (k > 0): exponential growth
- If (k < 0): exponential decay

Example:
Bacteria multiplying in a controlled environment.

1.2.2 Logistic Growth Model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

- $P(t)$: population at time t
- r: intrinsic growth rate

- K: carrying capacity

Explanation:
The logistic model describes population growth when resources are limited, so growth cannot continue indefinitely.

Key idea:
- When P is small $\rightarrow$ growth is fast (almost exponential)
- As P increases $\rightarrow$ growth slows down
- When $P = K \rightarrow$ growth stops

Behavior:
- S-shaped (sigmoid) growth curve
- Population stabilizes at K

Example:
Animal populations with limited food and space.

1.2.3 Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - T_{\text{env}})$$

- T: temperature of the object
- T_{env}: surrounding temperature
- $k > 0$: cooling constant

Explanation:
The rate of cooling depends on how different the object's temperature is from the environment.

Key idea:
- Large temperature difference $\rightarrow$ fast cooling
- Small difference $\rightarrow$ slow cooling

Example:
A hot cup of tea cooling in a room.

1.2.4 Radioactive Decay

$$\frac{dN}{dt} = -\lambda N$$

- $N(t)$: number of radioactive atoms
- $\lambda > 0$: decay constant

Explanation:
The rate of decay depends on how many atoms are still present.

Behavior:
- Fast at first, then slows down
- Never reaches zero exactly

Example:
Carbon dating.

1.2.5 Simple Harmonic Motion (Spring-Mass System)

$$m\frac{d^2x}{dt^2} + kx = 0$$

- $x(t)$: displacement
- m: mass
- k: spring constant

Explanation:
A restoring force pulls the object back toward equilibrium.

Key idea:
- Larger displacement $\rightarrow$ stronger restoring force

Behavior:
- Oscillatory (back and forth)
- Periodic motion

Example:
A mass on a spring.

1.3 Classification of Differential Equations

1.3.1 Ordinary vs. Partial Differential Equations

Type	Description	Example
ODE (Ordinary DE)	Involves derivatives with respect to **one** independent variable	$\frac{dy}{dx} + 2y = x$
PDE (Partial DE)	Involves **partial** derivatives with respect to **two or more** independent variables	$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

This course focuses entirely on ODEs.

1.4 Order, Degree, and Linearity

1.4.1 Order

The **order** of a differential equation is the order of the **highest derivative** present.

$$y'' + 3y' - 4y = 0 \qquad \Rightarrow \text{Order} = 2$$

$$\frac{d^3y}{dx^3} + x^2y = \sin x \qquad \Rightarrow \text{Order} = 3$$

1.4.2 Degree

The **degree** of a differential equation is the **power (exponent)** of the highest-order derivative, *after* the equation has been made rational and cleared of fractions.

$$\left(\frac{d^2y}{dx^2}\right)^3 + 4y = 0 \qquad \Rightarrow \text{Degree} = 3$$

Note: Not all ODEs have a defined degree (e.g., $e^{y''} + y = 0$).

1.4.3 Linearity

An nth-order ODE is **linear** if it can be written in the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Conditions for linearity:

1. The dependent variable y and all its derivatives appear to the **first power**.
2. No **products** of y with its derivatives.
3. Coefficients $a_i(x)$ depend only on x, **not** on y.

Equation	Linear?	Reason
$y'' + 4y' + 3y = 0$	Yes	Meets all conditions
$y'' + yy' = 0$	No	Product of y and y'
$y'' + \sin(y) = 0$	No	$\sin(y)$ is nonlinear in y
$y'' + x^2y = e^x$	Yes	Meets all conditions

1.5 Solutions of Ordinary Differential Equations

1.5.1 Definition of a Solution

Definition 1.1 — Solution of an ODE

A function $\phi(x)$, defined on an interval I, is called a **solution** of the ODE

$$F\big(x,\, y,\, y',\, y'',\, \dots,\, y^{(n)}\big) = 0$$

if:

1. $\phi(x)$ is n-times differentiable on I, and
2. The equation is **satisfied identically** on I when $y = \phi(x)$, that is:

$$F\big(x,\, \phi(x),\, \phi'(x),\, \phi''(x),\, \dots,\, \phi^{(n)}(x)\big) = 0 \quad \text{for all } x \in I$$

In other words, a solution is not just any function — it must be sufficiently differentiable **and** make the differential equation an identity (true for every value of x in the interval, not just at one point).

> **Key Insight:** Unlike algebraic equations whose solutions are *numbers*, the solutions of differential equations are *functions*.

1.5.2 Interval of Definition

A solution $\phi(x)$ is always defined on some interval $I = (a, b)$, which may be:

- A finite interval: $I = (a, b)$
- A semi-infinite interval: $I = (a, \infty)$ or $I = (-\infty, b)$
- The entire real line: $I = (-\infty, \infty)$

The interval must be such that the solution remains valid (i.e., no undefined expressions, no discontinuities of the solution or the ODE coefficients).

Example: The solution $y = \dfrac{1}{x}$ of the ODE $xy' + y = 0$ is valid on $(0, \infty)$ or $(-\infty, 0)$, but NOT on any interval containing $x = 0$.

1.5.3 Types of Solutions

1.5.3.1 1. General Solution

The **general solution** of an nth-order ODE contains exactly n **arbitrary constants** $C_1, C_2, \dots, C_n$. It represents a **family of curves** (one curve for each choice of constants). That is

$$y = f(x, C)$$

Geometric interpretation: Each value of the constant(s) gives a different solution curve (many solutions). Together, they fill the plane without crossing (in the regular case).

Example 1:

Consider the ODE $\frac{dy}{dx} = y$. The general solution is $y = Ce^x$.

Each value of C gives a different solution curve (exponential). Together, they form a **one-parameter family of curves**:

Value of C	Solution
$C = 1$	$y = e^x$
$C = -1$	$y = -e^x$
$C = 0$	$y = 0$ (trivial solution)
$C = 2$	$y = 2e^x$

No two of these curves ever intersect (each point in the plane lies on exactly one curve), which reflects the uniqueness of solutions.

Example 2: The general solution of $\frac{dy}{dx} = 2x$ is:

$$y = x^2 + C, \quad C \in \mathbb{R}$$

This is a family of upward-opening parabolas, all shifted vertically.

Example 3: The general solution of $y'' + y = 0$ is:

$$y = C_1 \cos x + C_2 \sin x$$

which contains **two** arbitrary constants (matching the order 2).

1.5.3.2 2. Particular Solution

A **particular solution** is obtained from the general solution by assigning **specific numerical values** to the arbitrary constants, typically by applying **initial conditions** or **boundary conditions**.

Example: Given $y = x^2 + C$ and the condition $y(1) = 5$:

$$5 = 1^2 + C \quad \Rightarrow \quad C = 4$$

So the particular solution is $y = x^2 + 4$.

1.5.3.3 3. Trivial Solution

The **trivial solution** is $y \equiv 0$ (identically zero). It always satisfies any **linear homogeneous** ODE, but it is often of little practical interest.

$$y'' + 4y' + 3y = 0 \quad \Rightarrow \quad y = 0 \text{ is always a solution}$$

1.5.4 Explicit and Implicit Solutions

Explicit solution: y is expressed directly as a function of x:

$$y = \phi(x) \qquad \text{e.g., } y = e^{3x} + \sin x$$

Implicit solution: The relationship between y and x is given by an equation $G(x, y) = 0$, and y cannot (or need not) be isolated:

$$x^2 + y^2 = 25 \qquad (\text{defines } y = \pm\sqrt{25 - x^2})$$

Even if we cannot solve explicitly for y, as long as $G(x, y) = 0$ implicitly defines a differentiable function that satisfies the ODE, it is a valid (implicit) solution.

1.5.5 Verification of a Solution

To **verify** (check) that $y = \phi(x)$ is a solution of a given ODE, follow these steps:

Step 1: Compute all derivatives of $\phi(x)$ needed by the equation.

Step 2: Substitute $y = \phi(x)$ and its derivatives into the ODE.

Step 3: Simplify both sides and confirm the equation holds **identically** (i.e., LHS = RHS for all x in the domain).

Example 1: Verify that $y = e^{-2x}$ is a solution of $y' + 2y = 0$.

Step 1: $y' = -2e^{-2x}$

Step 2: Substitute:

$$y' + 2y = -2e^{-2x} + 2e^{-2x} = 0\checkmark$$

The equation is satisfied identically. $y = e^{-2x}$ is a solution on $(-\infty, \infty)$.

Example 2: Verify that $y = x^2 + \dfrac{1}{x}$ is a solution of $x^2y'' - 2y = -2x^2$.

Step 1: Compute derivatives:

$$y' = 2x - \frac{1}{x^2}, \qquad y'' = 2 + \frac{2}{x^3}$$

Step 2: Substitute into LHS:

$$x^2y'' - 2y = x^2\left(2 + \frac{2}{x^3}\right) - 2\left(x^2 + \frac{1}{x}\right)$$

$$= 2x^2 + \frac{2}{x} - 2x^2 - \frac{2}{x} = 0$$

But the RHS $= -2x^2 \neq 0$ in general. So let us recheck... actually this is $\neq$ RHS, showing that not every guess works. A correct verification must confirm LHS = RHS.

Example 3: Show that $y = C_1e^{3x} + C_2e^{-x}$ is the general solution of $y'' - 2y' - 3y = 0$.

Step 1: Compute derivatives:

$$y' = 3C_1e^{3x} - C_2e^{-x}$$

$$y'' = 9C_1e^{3x} + C_2e^{-x}$$

Step 2: Substitute:

$$y'' - 2y' - 3y = (9C_1e^{3x} + C_2e^{-x}) - 2(3C_1e^{3x} - C_2e^{-x}) - 3(C_1e^{3x} + C_2e^{-x})$$

$$= e^{3x}(9C_1 - 6C_1 - 3C_1) + e^{-x}(C_2 + 2C_2 - 3C_2)$$

$$= e^{3x}(0) + e^{-x}(0) = 0\checkmark$$

This confirms $y = C_1e^{3x} + C_2e^{-x}$ is indeed a solution for **all** values of C_1 and C_2.

Example 4: Verify that $y = \sin(2x)$ is a solution of $y'' + 4y = 0$.

Step 1:

$$y' = 2\cos(2x), \qquad y'' = -4\sin(2x)$$

Step 2:

$$y'' + 4y = -4\sin(2x) + 4\sin(2x) = 0\checkmark$$

Step 3: Note that $y = \cos(2x)$ is **also** a solution:

$$y' = -2\sin(2x), \quad y'' = -4\cos(2x)$$

$$y'' + 4y = -4\cos(2x) + 4\cos(2x) = 0\checkmark$$

And so is $y = C_1 \sin(2x) + C_2 \cos(2x)$ for any constants C_1, C_2 — this is the **general solution**.

Example 5 (Implicit Solution): Verify that $x^2 + y^2 = 25$ is an implicit solution of $y\,dy + x\,dx = 0$.

Step 1: Differentiate $x^2 + y^2 = 25$ implicitly:

$$2x + 2y\frac{dy}{dx} = 0$$

Step 2: Rearrange:

$$y\,dy + x\,dx = 0 \checkmark$$

The implicit relation satisfies the ODE identically.

Example 6 (Particular Solution): Given the ODE $\frac{dy}{dx} = 3x^2$ with initial condition $y(0) = 2$:

The general solution is:

$$y = \int 3x^2\,dx = x^3 + C$$

Applying $y(0) = 2$:

$$2 = 0 + C \quad \Rightarrow \quad C = 2$$

Particular solution: $\boxed{y = x^3 + 2}$

Summary — Types of Solutions at a Glance

Type	Description	How obtained
General solution	Contains n arbitrary constants for an nth-order ODE	Solve the ODE without extra conditions
Particular solution	Specific solution with constants determined	Apply initial or boundary conditions
Trivial solution	$y \equiv 0$	Always satisfies homogeneous linear ODEs
Explicit solution	y expressed directly as $y = \phi(x)$	Direct method
Implicit solution	$G(x, y) = 0$ defines y implicitly	Integration of exact equations