MAT 102: Ordinary Differential Equations

Topic 1: Introduction to Differential Equations

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Tutorial Questions — Topic 1

Instructions

Work through all questions below. Show all steps clearly. Questions are graded from routine to challenging.

Section A: Classification (Order, Degree, Linearity)

Q1. For each of the following ODEs, state: (i) the **order**, (ii) the **degree**, and (iii) whether it is **linear or nonlinear**. Give a reason for (iii).

#	Equation
(a)	$\frac{dy}{dx} - 5y = e^x$
(b)	$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 7y = 0$
(c)	$\left(\frac{dy}{dx}\right)^2 + y = x^3$
(d)	$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$
(e)	$\frac{d^3y}{dx^3} + x^2\frac{dy}{dx} - (\sin x)y = \cos x$

#	Equation
(f)	$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}$
(g)	$e^{y''} - y = 0$
(h)	$\frac{d^4y}{dx^4} - 16y = 0$

Q2. Classify each of the following as ODE or PDE, and for ODEs, state the order, degree, and linearity:

(a) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (Wave equation)

(b) $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$ (Pendulum)

(c) $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (Heat equation)

(d) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ (Legendre's equation)

(e) $x^2y'' + xy' + (x^2 - n^2)y = 0$ (Bessel's equation)

Section B: Verification of Solutions

Q3. Verify that the given function is a solution of the corresponding ODE on the stated interval.

(a) $y = 3e^{2x}$; ODE: $y' - 2y = 0$; Interval: $(-\infty, \infty)$

(b) $y = x^3$; ODE: $xy' - 3y = 0$; Interval: $(0, \infty)$

(c) $y = e^x \sin x$; ODE: $y'' - 2y' + 2y = 0$; Interval: $(-\infty, \infty)$

(d) $y = x^2 \ln x$; ODE: $x^2y'' - 3xy' + 4y = 0$; Interval: $(0, \infty)$

(e) $y = C_1e^x + C_2e^{-2x}$; ODE: $y'' + y' - 2y = 0$

(f) $y = \frac{1}{1-x}$; ODE: $y' = y^2$; Interval: $(-\infty, 1)$

(g) $y = x\cos(\ln x)$; ODE: $x^2y'' - xy' + 2y = 0$; Interval: $(0, \infty)$

(Hint for (g): compute y' and y'' carefully using the product and chain rules.)

Q4. Determine the value(s) of k such that $y = e^{kx}$ is a solution of each ODE:

(a) $y'' - 5y' + 6y = 0$

(b) $y'' + ky' - 2ky = 0$

(c) $4y'' - y = 0$

Section C: General and Particular Solutions

Q5. For each ODE, (i) find the general solution, and (ii) find the particular solution satisfying the given initial condition.

(a) $\frac{dy}{dx} = 4x^3 - 1; \quad y(0) = 3$

(b) $\frac{dy}{dx} = \frac{x}{y}; \quad y(2) = 3$

(c) $\frac{dy}{dx} = e^{2x}; \quad y(0) = \frac{1}{2}$

(d) $\frac{d^2y}{dx^2} = 6x; \quad y(0) = 1,\ y'(0) = 2$

(e) $\frac{dy}{dx} = 3y; \quad y(0) = 5$

Q6. Determine whether the following functions are solutions of $y'' - y = 0$:

(a) $y = \cosh x$

(b) $y = \sinh x$

(c) $y = e^x + e^{-x}$

(d) $y = Ae^x + Be^{-x}$ for arbitrary A, B

Which of these represent general solutions? Which are particular solutions? Explain.

Section D: Mathematical Models — Setting Up ODEs

Q7. In each case, **set up** (but do not solve) the differential equation that models the given situation. Clearly define all variables and constants used.

(a) A population of bacteria grows at a rate proportional to the current population.

(b) A population grows rapidly at first but slows as it approaches a carrying capacity due to limited resources.

(c) A cup of tea at temperature $T_0 = 95°C$ is placed in a room at $25°C$. The rate of cooling is proportional to the excess temperature above the room temperature.

(d) A radioactive substance decays at a rate proportional to the amount present. The half-life is $T_{1/2}$.

(e) A body of mass m falls under gravity with air resistance proportional to its velocity v.
