MAT 102: Ordinary Differential Equations

Topic 2: First Order Differential Equations

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Table of contents

1 Linear First Order Equations

1.1 Definition and Standard Form

A first-order ODE is **linear** if it can be written as:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x only (not of y).

- If $Q(x) = 0$: the equation is **homogeneous linear** → separable.
- If $Q(x) \neq 0$: the equation is **non-homogeneous linear**.

1.2 Method of Integrating Factor

The key idea: multiply through by a special function $\mu(x)$ so that the left-hand side becomes an exact derivative $\frac{d}{dx}[\mu y]$.

Algorithm — Integrating Factor Method

Step 1. Write the equation in standard form: $\frac{dy}{dx} + P(x)y = Q(x)$
Step 2. Compute $\mu(x) = e^{\int P(x)\,dx}$ (no constant of integration needed here)
Step 3. Multiply both sides by $\mu(x)$:

$$\frac{d}{dx}[\mu(x)\,y] = \mu(x)Q(x)$$

Step 4. Integrate both sides:

$$\mu(x)\,y = \int \mu(x)Q(x)\,dx + C$$

Step 5. Solve for y:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right]$$

Why does this work? Notice that $\mu'(x) = \mu(x)P(x)$, so:

$$\mu y' + \mu P(x)y = \frac{d}{dx}[\mu y]$$

This is exactly the product rule applied to μy.

Example 1: Solve $\frac{dy}{dx} - 2y = 4x$.

Standard form: $P(x) = -2$, $Q(x) = 4x$.

$$\mu = e^{\int -2\,dx} = e^{-2x}$$

$$\frac{d}{dx}[e^{-2x}y] = 4xe^{-2x}$$

$$e^{-2x}y = \int 4xe^{-2x}\,dx$$

Using integration by parts ($u = 4x$, $dv = e^{-2x}dx$):

$$= -2xe^{-2x} + \int 2e^{-2x}\,dx = -2xe^{-2x} - e^{-2x} + C$$

$$\boxed{y = -2x - 1 + Ce^{2x}}$$

Example 2: Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2, \quad x > 0.$

$P(x) = \dfrac{1}{x}$, $Q(x) = x^2$.

$$\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

$$\frac{d}{dx}[xy] = x \cdot x^2 = x^3$$

$$xy = \int x^3\,dx = \frac{x^4}{4} + C$$

$$\boxed{y = \frac{x^3}{4} + \frac{C}{x}}$$

Example 3: Solve $\dfrac{dy}{dx} + 2xy = 2x.$

$P(x) = 2x$, $Q(x) = 2x$.

$$\mu = e^{\int 2x\,dx} = e^{x^2}$$

$$\frac{d}{dx}\left[e^{x^2}y\right] = 2xe^{x^2}$$

$$e^{x^2}y = \int 2xe^{x^2}\,dx = e^{x^2} + C$$

$$\boxed{y = 1 + Ce^{-x^2}}$$

Note: as $x \to \infty$, $y \to 1$ for any C — the equilibrium solution is $y = 1$.

Example 4: Solve $x\dfrac{dy}{dx} - 3y = x^4, \quad x > 0.$

Divide by x: $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3$.

$P(x) = -\dfrac{3}{x}$, $Q(x) = x^3$.

$$\mu = e^{\int -3/x\,dx} = e^{-3\ln x} = x^{-3}$$

$$\frac{d}{dx}[x^{-3}y] = x^{-3} \cdot x^3 = 1$$

$$x^{-3}y = x + C \quad \Rightarrow \quad \boxed{y = x^4 + Cx^3}$$

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2 Separable Equations

2.1 Definition

A first-order ODE is **separable** if it can be written so that all terms involving y (and dy) are on one side, and all terms involving x (and dx) are on the other:

$$\frac{dy}{dx} = f(x)\,g(y) \quad \Longleftrightarrow \quad \frac{dy}{g(y)} = f(x)\,dx \quad (g(y) \neq 0)$$

Algorithm — Separation of Variables

Step 1. Rewrite as $\dfrac{dy}{g(y)} = f(x)\,dx$

Step 2. Integrate both sides: $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx + C$

Step 3. Solve for y explicitly if possible, otherwise leave implicitly.

Step 4. Check for any solutions lost when dividing by $g(y)$ (i.e., where $g(y) = 0$).

Example 1: Solve $\frac{dy}{dx} = xy$.

$$\frac{dy}{y} = x\,dx$$

$$\int \frac{dy}{y} = \int x\,dx \quad \Rightarrow \quad \ln|y| = \frac{x^2}{2} + C_1$$

$$|y| = e^{C_1} e^{x^2/2} \quad \Rightarrow \quad \boxed{y = Ae^{x^2/2}}, \quad A \in \mathbb{R}$$

(Note: $A = 0$ gives the trivial solution $y = 0$, which was lost when dividing by y.)

Example 2: Solve $\frac{dy}{dx} = \frac{x^2}{1-y^2}$.

$$(1-y^2)\,dy = x^2\,dx$$

$$\int (1-y^2)\,dy = \int x^2\,dx$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + C$$

$$\boxed{3y - y^3 = x^3 + C} \qquad \text{(implicit solution)}$$

Example 3: Solve $(1+x)\,dy - y\,dx = 0$.

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + C_1$$

$$\boxed{y = A(1+x)}, \quad A \neq 0$$

Example 4: Solve $\dfrac{dy}{dx} = e^{x-y}$ with $y(0) = 0$.

$$\frac{dy}{dx} = e^x \cdot e^{-y}$$

$$e^y\,dy = e^x\,dx$$

$$e^y = e^x + C$$

Apply $y(0) = 0$: $e^0 = e^0 + C \Rightarrow C = 0$.

$$e^y = e^x \quad \Rightarrow \quad \boxed{y = x}$$

Example 5: Radioactive decay. A substance decays at rate proportional to amount present. If initially N_0 grams, find $N(t)$.

$$\frac{dN}{dt} = -\lambda N \quad \Rightarrow \quad \frac{dN}{N} = -\lambda\,dt$$

$$\ln N = -\lambda t + C \quad \Rightarrow \quad \boxed{N(t) = N_0 e^{-\lambda t}}$$

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3 Homogeneous Differential Equations

3.1 Definition

A first-order ODE $\dfrac{dy}{dx} = f(x, y)$ is called **homogeneous** if $f(x, y)$ can be expressed purely as a function of the ratio $\dfrac{y}{x}$:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

This means the right side depends only on the ratio y/x, not on x and y separately

Test: An ODE $M\,dx + N\,dy = 0$ is homogeneous if M and N are both **homogeneous functions of the same degree**, i.e., $M(tx, ty) = t^n M(x, y)$ and $N(tx, ty) = t^n N(x, y)$.

$M\,dx + N\,dy = 0$ is considered **homogeneous** if you can scale both variables by a factor t and "pull out" the same power of   from both functions

Algorithm — Homogeneous Substitution

Step 1. Confirm the equation is homogeneous, that is check if the equation can be written as:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Step 2. Use substitution. Let $v = \dfrac{y}{x}$, so $y = vx$
Then differentiate:
$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.
Step 3. Substitute into original equation to get a separable equation in v and x.
Step 4. Separate and integrate.
Step 5. Back-substitute $v = y/x$.

Example 1: Solve $\dfrac{dy}{dx} = \dfrac{x+y}{x}$

step1: $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}$ ....(1) (homogeneous)

Step2: Let $v = \frac{y}{x}$ implies $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

step3: Replace $y = vx$, $y/x = v$, and $dy/dx = v + xdv/dx$ into ....(1) to get

$$v + x\frac{dv}{dx} = 1 + v$$

Simplify

$$\frac{dv}{dx} = \frac{1}{x}$$

Separate variables and integrate both sides:

$$\int dv = \int \frac{1}{x}dx$$

$$v = \ln|x| + C$$

Replace $v = y/x$ and simplify:

$$y = x(\ln|x| + C)$$

Example 2: Solve $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}$.

Let $v = y/x$, $y = vx$:

$$v + x\frac{dv}{dx} = v + \frac{1}{v}$$

$$x\frac{dv}{dx} = \frac{1}{v}$$

$$v\,dv = \frac{dx}{x}$$

$$\frac{v^2}{2} = \ln|x| + C$$

$$\frac{y^2}{2x^2} = \ln|x| + C \quad \Rightarrow \quad \boxed{y^2 = 2x^2(\ln|x| + C)}$$

Example 3: Solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$.

Let $v = y/x$:

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = \frac{-v^2 - 1}{2v}$$

$$\frac{2v\,dv}{v^2 + 1} = -\frac{dx}{x}$$

$$\ln(v^2 + 1) = -\ln|x| + C$$

$$(v^2 + 1)x = A \quad \Rightarrow \quad \boxed{y^2 + x^2 = Ax}$$

Example 4: Solve $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

Check: $M = x^2 + y^2$ (degree 2), $N = -2xy$ (degree 2). Homogeneous.

Let $v = y/x$:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} = \frac{1 + v^2}{2v}$$

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1 + v^2 - 2v^2}{2v} = \frac{1 - v^2}{2v}$$

$$\frac{2v\,dv}{1 - v^2} = \frac{dx}{x}$$

$$-\ln|1 - v^2| = \ln|x| + C_1$$

$$(1 - v^2)x = A \quad \Rightarrow \quad x\left(1 - \frac{y^2}{x^2}\right) = A \quad \Rightarrow \quad \boxed{x^2 - y^2 = Ax}$$

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4 Bernoulli Equation

4.1 Definition

The **Bernoulli equation** is:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \in \mathbb{R} \neq 0, 1$$

- When $n = 0$: reduces to a linear equation.
- When $n = 1$: reduces to a separable equation.
- For $n \neq 0, 1$: requires the Bernoulli substitution.

Algorithm — Bernoulli Substitution

Step 1. Divide both sides by y^n:

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

Step 2. Let $v = y^{1-n}$. Then $\dfrac{dv}{dx} = (1 - n)y^{-n}\dfrac{dy}{dx}$, so $y^{-n}\dfrac{dy}{dx} = \dfrac{1}{1 - n}\dfrac{dv}{dx}$.

Step 3. The equation becomes **linear** in v:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Step 4. Solve using the integrating factor method.
Step 5. Back-substitute $v = y^{1-n}$.

Example 1: Solve $\dfrac{dy}{dx} + y = xy^3$.

$n = 3$, $P = 1$, $Q = x$. Divide by y^3:

$$y^{-3}\frac{dy}{dx} + y^{-2} = x ...(*)$$

Use Substitution, let $v = y^{1-3} = y^{-2}$, $y^{-3}dy/dx = -1/2dv/dx$. Substitute to ...(*)

$$\frac{dv}{dx} - 2v = -2x$$

Integrating factor: $\mu = e^{-2x}$

$$\frac{d}{dx}[e^{-2x}v] = -2xe^{-2x}$$

$$e^{-2x}v = \int -2xe^{-2x}\,dx$$

$$\text{Let } u = -2x \Rightarrow du = -2dx. \quad \text{Let } dv = e^{-2x}dx \Rightarrow v = \int e^{-2x}dx = \frac{1}{2}e^{-2x}$$

$$e^{-2x}v = \int -2xe^{-2x}\,dx = xe^{-2x} + \frac{1}{2}e^{-2x} + C$$

$$v = x + \frac{1}{2} + Ce^{2x} \quad \Rightarrow \quad \boxed{y^{-2} = x + \frac{1}{2} + Ce^{2x}}$$

Example 2: Solve $\dfrac{dy}{dx} - \dfrac{y}{x} = xy^2$.

$n = 2$, $P = -1/x$, $Q = x$. Let $v = y^{-1}$:

$$\frac{dv}{dx} + \frac{v}{x} = -x$$

$\mu = e^{\int 1/x\,dx} = x$

$$\frac{d}{dx}[xv] = -x^2$$

$$xv = -\frac{x^3}{3} + C \quad \Rightarrow \quad v = -\frac{x^2}{3} + \frac{C}{x}$$

$$\boxed{\frac{1}{y} = \frac{C}{x} - \frac{x^2}{3}}$$

Example 3: Solve $3\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{12}{y^2}, \quad x > 0$.

Rewrite: $\dfrac{dy}{dx} + \dfrac{y}{3x} = \dfrac{4}{y^2}$.

$n = -2$, so $v = y^{1-(-2)} = y^3$, $\dfrac{dv}{dx} = 3y^2\dfrac{dy}{dx}$.

Multiply original by $3y^2$:

$$3y^2\frac{dy}{dx} + \frac{y^3}{x} = 12 \quad \Rightarrow \quad \frac{dv}{dx} + \frac{v}{x} = 12$$

$\mu = x$:

$$\frac{d}{dx}[xv] = 12x \quad \Rightarrow \quad xv = 6x^2 + C$$

$$\boxed{y^3 = 6x + \frac{C}{x}}$$

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5 Exact Differential Equations and Integrating Factors

Exact Equation

The equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is **exact** on a region R if there exists a function $F(x,y)$ such that:

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

The general solution is then $F(x,y) = C$.

Test for Exactness: If M, N, M_y, N_x are continuous on R, then the equation is exact if and only if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Why? If F exists, then $F_{xy} = M_y$ and $F_{yx} = N_x$. By equality of mixed partials (Clairaut's theorem), $M_y = N_x$.

Algorithm — Solving Exact Equations

Step 1. Check $M_y = N_x$. If not equal → not exact (try integrating factor).
Step 2. Integrate M with respect to x:

$$F(x,y) = \int M(x,y)\,dx + h(y)$$

Step 3. Differentiate F with respect to y and set equal to N:

$$\frac{\partial F}{\partial y} = N(x,y) \quad \Rightarrow \quad \text{solve for } h'(y)$$

Step 4. Integrate $h'(y)$ to find $h(y)$.
Step 5. Write the general solution: $F(x,y) = C$.

Example 1: Solve $(2xy+3)\,dx + (x^2+4y)\,dy = 0$. **Step 1: Identify** $M = 2xy+3$, $N = x^2+4y$.

Step 2: Check exactness (Differentiate) $M_y = 2x$, $N_x = 2x$. Exact.

Step 3: Integrate M with respect to x, treat y as constant and add unknown function of y

$$F = \int (2xy + 3)\,dx = x^2y + 3x + h(y)$$

Step 4: Differentiate F wrt y and set equal to N to get the unknown function of y.

$$F_y = x^2 + h'(y) = x^2 + 4y \quad \Rightarrow \quad h'(y) = 4y$$

Step 5: Integrate to get the unknown function

$$h = 2y^2$$

Final answer:

$$F = \boxed{x^2y + 3x + 2y^2 = C}$$

>If the differential of a function is zero, then the function is not changing. So its value stays constant. Hence: $F(x,y) = C$, where C is a constant.

Example 2: Solve $(e^x \sin y - 2y \sin x)\,dx + (e^x \cos y + 2\cos x)\,dy = 0$.

$M = e^x \sin y - 2y \sin x$, $N = e^x \cos y + 2\cos x$.

$M_y = e^x \cos y - 2\sin x$, $N_x = e^x \cos y - 2\sin x$. Exact.

$$F = \int M\,dx = e^x \sin y + 2y \cos x + h(y)$$

$$F_y = e^x \cos y + 2\cos x + h'(y) = e^x \cos y + 2\cos x \quad \Rightarrow \quad h'(y) = 0 \quad \Rightarrow \quad h = 0$$

$$\boxed{e^x \sin y + 2y \cos x = C}$$

Example 3: Solve $(3x^2y + 2xy^2)\,dx + (x^3 + 2x^2y)\,dy = 0$.

$M_y = 3x^2 + 4xy$, $N_x = 3x^2 + 4xy$. Exact.

$$F = \int (3x^2y + 2xy^2)\,dx = x^3y + x^2y^2 + h(y)$$

$$F_y = x^3 + 2x^2y + h'(y) = x^3 + 2x^2y \quad \Rightarrow \quad h'(y) = 0$$

$$\boxed{x^3y + x^2y^2 = C}$$

5.1 Integrating Factors for Non-Exact Equations

If $M_y \neq N_x$, multiply through by an **integrating factor** μ to make the equation exact.

Finding μ:

$$\text{If} \quad \frac{M_y - N_x}{N} = \phi(x) \text{ (function of } x \text{ only)} \quad \Rightarrow \quad \mu(x) = e^{\int \phi(x)\,dx}$$

$$\text{If} \quad \frac{N_x - M_y}{M} = \psi(y) \text{ (function of } y \text{ only)} \quad \Rightarrow \quad \mu(y) = e^{\int \psi(y)\,dy}$$

Example 4: Solve $(y^2 - x)\,dx + x\,dy = 0$ using an integrating factor.

$M = y^2 - x$, $N = x$. $M_y = 2y$, $N_x = 1$. Not exact.

$$\frac{M_y - N_x}{N} = \frac{2y - 1}{x} \qquad \text{not a function of } x \text{ alone}$$

$$\frac{N_x - M_y}{M} = \frac{1 - 2y}{y^2 - x} \qquad \text{not clean}$$

Try Another Method, rewrite as:

$$x\frac{dy}{dx} = x - y^2 \quad \Rightarrow \quad \frac{dy}{dx} = 1 - \frac{y^2}{x} \quad \text{— Bernoulli with } n = 2$$

Substitute and solve (linear in v)

Let $v = y^{-1}$: $\frac{dv}{dx} + \frac{v}{x} = -\frac{1}{x}$... (linear in v).

This illustrates that **some equations can be solved by multiple approaches**.

Key Idea You Must Understand When:

$$M_y \neq N_x$$

The equation is NOT exact So you must: Either - Find an integrating factor, Or

- Transform into another type:
  - Bernoulli, Homogeneous, Linear

Example 5 : Solve $(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$.

$M = 2y - 6x$, $N = 3x - 4x^2y^{-1}$.

$M_y = 2$, $N_x = 3 - 8xy^{-1}$. Not exact.

$$\frac{M_y - N_x}{N} = \frac{2 - 3 + 8xy^{-1}}{3x - 4x^2y^{-1}} = \frac{-1 + 8xy^{-1}}{x(3 - 4xy^{-1})}$$

This is messy. Instead, try $\frac{N_x - M_y}{M}$:

Sometimes trial and error with $\mu = x^a y^b$ is needed. This is a topic for further study.

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6 Initial Value Problems

Definition — Initial Value Problem (IVP)

An **Initial Value Problem** for a first-order ODE consists of:

1. The ODE: $\dfrac{dy}{dx} = f(x, y)$
2. An **initial condition**: $y(x_0) = y_0$

The initial condition selects **one** particular solution from the family of general solutions.

6.1 Geometric Meaning

The initial condition $y(x_0) = y_0$ means the solution curve must pass through the point (x_0, y_0) in the xy-plane.

Example (Separable Equation) Solve: $\frac{dy}{dx} = xy, \quad y(0) = 3$ Solution:
$\frac{dy}{y} = xdx \Rightarrow \int \frac{1}{y}dy = \int xdx \Rightarrow \ln|y| = \frac{x^2}{2} + C$
$Y = Ce^{x^2/2}, \quad$ Apply initial condition $, C = 3$ $y = 3e^{x^2/2}$
Example 1: Solve $y' + 2y = 4, \quad y(0) = 1$.
$P = 2$, $Q = 4$. $\mu = e^{2x}$.
$\frac{d}{dx}[e^{2x}y] = 4e^{2x}$
$e^{2x}y = 2e^{2x} + C \quad \Rightarrow \quad y = 2 + Ce^{-2x}$
Apply $y(0) = 1$: $1 = 2 + C \Rightarrow C = -1$.
$\boxed{y = 2 - e^{-2x}}$

Example 2: Solve $\dfrac{dy}{dx} = 2x(1 + y^2), \quad y(0) = 0$.

Separable: $\dfrac{dy}{1+y^2} = 2x\,dx$

$$\arctan y = x^2 + C$$

Apply $y(0) = 0$: $\arctan 0 = 0 + C \Rightarrow C = 0$.

$$\arctan y = x^2 \quad \Rightarrow \quad \boxed{y = \tan(x^2)}$$

Example 3: Solve $xy' + y = e^x, \quad y(1) = 2$.

Divide by x: $y' + \dfrac{y}{x} = \dfrac{e^x}{x}$. $\mu = x$.

$$\frac{d}{dx}[xy] = e^x \quad \Rightarrow \quad xy = e^x + C$$

Apply $y(1) = 2$: $1 \cdot 2 = e + C \Rightarrow C = 2 - e$.

$$\boxed{y = \frac{e^x + 2 - e}{x}}$$

6.2 Existence and Uniqueness Theorem

Theorem (Picard–Lindelöf)

Consider the IVP:

$$y' = f(x, y), \quad y(x_0) = y_0$$

If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are **continuous** on some open rectangle R containing (x_0, y_0), then there exists a unique solution $y = \phi(x)$ on some interval $|x - x_0| < h$ for some $h > 0$.

Interpretation:

- **Existence:** a solution *does* exist (no dead ends).
- **Uniqueness:** only *one* solution passes through (x_0, y_0) — solution curves cannot cross.
- The theorem gives no formula — it is a theoretical guarantee.

What can go wrong?

Situation	Consequence
f discontinuous at (x_0, y_0)	May have no solution
$\partial f/\partial y$ discontinuous	May have infinitely many solutions
Solution exists but is unbounded	Interval of existence may be finite

Example: The IVP $y' = y^{2/3}$, $y(0) = 0$.

Here $f = y^{2/3}$ is continuous, but $\partial f/\partial y = \frac{2}{3}y^{-1/3}$ is **not** continuous at $y = 0$.

Solutions: $y = 0$ and $y = \left(\frac{x}{3}\right)^3$ **both** pass through the origin → **uniqueness fails**.

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7 Applications of First Order ODEs

7.1 Exponential Growth and Decay

$$\frac{dP}{dt} = kP, \quad P(0) = P_0 \quad \Rightarrow \quad P(t) = P_0 e^{kt}$$

$k > 0$	Growth	Population, investment
$k < 0$	Decay	Radioactivity, drug elimination

7.2 Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - T_{\text{env}}), \quad T(0) = T_0$$

Solution:

$$\boxed{T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-kt}}$$

Worked Example: A body found at $30°C$ in a room at $20°C$. After 1 hour, the body is $25°C$. Normal body temperature is $37°C$. When did death occur?

At $t = 0$ (time of finding): $T_0 = 30°C$, $T_\infty = 20°C$.

$$30 - 20 = (T_{\text{death}} - 20)e^{-k \cdot t_{\text{death}}}$$

At $t = 1$: $25 = 20 + 10e^{-k} \Rightarrow e^{-k} = 1/2 \Rightarrow k = \ln 2$.

At death: $37 = 20 + (30 - 20)e^{k\tau}$... More precisely, let τ = time before body was found:

$$37 - 20 = (30 - 20)e^{k\tau} \Rightarrow 17 = 10 \cdot 2^{\tau} \Rightarrow \tau = \log_2(1.7) \approx 0.766 \text{ hrs} \approx 46 \text{ min}$$

Death occurred approximately **46 minutes** before discovery.

7.3 Logistic Growth

When a population is limited by carrying capacity K:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0$$

Solution (by separation of variables + partial fractions):

$$\boxed{P(t) = \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right)e^{-rt}}}$$

As $t \to \infty$, $P \to K$ (population approaches carrying capacity).
