MAT 102: Ordinary Differential Equations

Topic 2: First Order Differential Equations

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Tutorial Questions — Topic 2

Instructions

Show all working clearly. Specify the method used for each equation.

Section A: Linear First Order Equations

Q1. Solve the following linear ODEs using the integrating factor method:

(a) $\dfrac{dy}{dx} + 3y = 6$

(b) $\dfrac{dy}{dx} - y = e^{2x}$

(c) $\dfrac{dy}{dx} + \dfrac{2y}{x} = x^3, \quad x > 0$

(d) $x\dfrac{dy}{dx} + 2y = x^2 - x + 1$

(e) $\dfrac{dy}{dx} + y\cos x = \sin x \cos x$

(f) $(1+x^2)\dfrac{dy}{dx} + 2xy = \dfrac{1}{1+x^2}$

Q2. Solve the following IVPs:

(a) $y' - 2y = 0, \quad y(0) = 3$

(b) $y' + y = 2e^{-x}, \quad y(0) = 1$

(c) $xy' - y = x^2 \sin x, \quad y(\pi) = 0$

(d) $y' + (\tan x)y = \cos^2 x, \quad y(0) = 1$

Section B: Separable Equations

Q3. Solve by separation of variables:

(a) $\dfrac{dy}{dx} = \dfrac{x^2}{y}$

(b) $\dfrac{dy}{dx} = (1+y^2)e^x$

(c) $\dfrac{dy}{dx} = \dfrac{y \ln y}{x}$

(d) $(1+x)\dfrac{dy}{dx} = 1 + y^2$

(e) $\dfrac{dy}{dx} = \sqrt{y}\,\sin x$

(f) $y(2x+3y^2)\,dx = x\,dy$ *(Hint: rearrange.)*

Q4. Solve the IVPs:

(a) $\dfrac{dy}{dx} = xy^2, \quad y(0) = 1$

(b) $\dfrac{dy}{dx} = \dfrac{y^2-1}{x}, \quad y(1) = 2$

(c) $\sin y \dfrac{dy}{dx} = \cos x(2\cos y - 1), \quad y(0) = \pi/2$

Section C: Homogeneous Equations

Q5. Show that each equation is homogeneous, then solve:

(a) $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

(b) $x^2 \dfrac{dy}{dx} = y^2 + xy$

(c) $(x^2 + 2xy)\, dx - x^2\, dy = 0$

(d) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$

(e) $(x^3 + y^3)\, dx - xy^2\, dy = 0$

Section D: Bernoulli Equations

Q6. Solve each Bernoulli equation:

(a) $\dfrac{dy}{dx} + y = y^2$

(b) $x \dfrac{dy}{dx} + y = x^4 y^3$

(c) $\dfrac{dy}{dx} = y(xy^3 - 1)$

(d) $\dfrac{dy}{dx} - 2y = -y^2 e^x$

(e) $2 \dfrac{dy}{dx} - y = xy^{-2}$

Section E: Exact Equations and Integrating Factors

Q7. Test for exactness. If exact, solve. If not, state why.

(a) $(2x + 3y)\, dx + (3x - y^2)\, dy = 0$

(b) $(ye^{xy} + 2xy)\, dx + (xe^{xy} + x^2)\, dy = 0$

(c) $(x^2 - y \sin x)\, dx + \cos x\, dy = 0$

(d) $(2xy - y^2)\, dx + (2xy - x^2)\, dy = 0$

(e) $(3x^2 y - 6x)\, dx + (x^3 + 2y)\, dy = 0$

Q8. Find an integrating factor and solve:

(a) $y\, dx + (2x - ye^y)\, dy = 0$

(b) $(x^2 + y^2 + x)\, dx + xy\, dy = 0$

Section F: Existence and Uniqueness

Q9. For each IVP, state whether the Picard–Lindelöf theorem guarantees a unique solution in a neighbourhood of the given point. If not, explain why.

(a) $y' = \sqrt{1 - y^2}, \quad y(0) = 0$

(b) $y' = x^2 + y^2, \quad y(0) = 1$

(c) $y' = |y|^{1/2}, \quad y(0) = 0$

(d) $y' = \dfrac{x}{y-1}, \quad y(0) = 1$

(e) $y' = \dfrac{1}{x-2} + y, \quad y(0) = 3$

Section G: Applications

Q10. A culture of bacteria doubles in 3 hours. Initially there are 500 bacteria.

(a) Write and solve the ODE governing the population $P(t)$.

(b) How many bacteria are there after 9 hours?

(c) When will the population reach 100,000?

Q11. A thermometer reading $-10°C$ is brought into a room at $20°C$. After 1 minute, the reading is $0°C$. When will the thermometer read $15°C$? Use Newton's law of cooling.

Q12. A tank contains 300 litres of water with 15 kg of dissolved salt. Salt solution of concentration 0.1 kg/litre flows in at 2 litres/min, and the well-mixed solution flows out at 2 litres/min. Find:

(a) The differential equation for $A(t)$, the amount of salt (kg) at time t.

(b) The solution $A(t)$.

(c) The amount of salt after 30 minutes.

(d) The limiting amount of salt as $t \to \infty$.

Q13. A body of mass 2 kg falls from rest. Air resistance equals $0.5v$ N, where v is the velocity in m/s. Use $g = 10$ m/s².

(a) Set up and solve the ODE for $v(t)$.

(b) Find the terminal velocity.

(c) Find the position $x(t)$ if $x(0) = 0$.

Q14 (Challenge). A population $P(t)$ follows logistic growth with $k = 0.4$, carrying capacity $M = 10{,}000$, and $P(0) = 500$.

(a) Write and solve the logistic equation.

(b) When will the population reach 5,000?

(c) Sketch the solution curve and describe the long-term behaviour.