MAT 102: Ordinary Differential Equations

Topic 3: Second Order Linear Ordinary Differential Equations

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1 Overview

Many things in nature move, vibrate, or change with time.
Examples include:

- a swinging pendulum,
- a vibrating spring,
- and electrical circuits.

These systems can often be described using **second-order linear differential equations**.

A **second-order linear ODE** involves the second derivative y'' as the highest derivative.

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2 General Theory of Second Order Linear ODEs

The **general second-order linear ODE** has the form:

$$y'' + P(x)y' + Q(x)y = R(x) \tag{3.1}$$

where $P(x)$, $Q(x)$, and $R(x)$ are functions of x alone.

Equation (3.1) is called **linear** because y, y', and y'' appear only to the first power.

- **Homogeneous** equation: $R(x) = 0 \Rightarrow y'' + P(x)y' + Q(x)y = 0$

- **Non-homogeneous** equation: $R(x) \neq 0$

The study of the general theory focuses on understanding:

- the structure of solutions,
- existence and uniqueness of solutions,
- homogeneous and nonhomogeneous equations,
- methods for finding solutions, etc

One important property of linear equations is the **superposition principle**, which states that linear combinations of solutions of the homogeneous equation are also solutions.

Superposition Principle

If (y_1) and (y_2) are solutions of the homogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

then any linear combination

$$y = C_1 y_1 + C_2 y_2$$

is also a solution, where (C_1) and (C_2) are constants.
Proof
Since (y_1) and (y_2) are solutions,

$$a(x)y_1'' + b(x)y_1' + c(x)y_1 = 0$$

and

$$a(x)y_2'' + b(x)y_2' + c(x)y_2 = 0.$$

Let

$$y = C_1 y_1 + C_2 y_2.$$

Differentiate:

$$y' = C_1 y_1' + C_2 y_2'$$

and

$$y'' = C_1 y_1'' + C_2 y_2''.$$

Substitute into the differential equation:

$$a(x)y'' + b(x)y' + c(x)y$$

$$= a(x)(C_1 y_1'' + C_2 y_2'') + b(x)(C_1 y_1' + C_2 y_2') + c(x)(C_1 y_1 + C_2 y_2).$$

Group the terms:

$$= C_1[a(x)y_1'' + b(x)y_1' + c(x)y_1] + C_2[a(x)y_2'' + b(x)y_2' + c(x)y_2].$$

Since (y_1) and (y_2) satisfy the homogeneous equation,

$$a(x)y_1'' + b(x)y_1' + c(x)y_1 = 0$$

and

$$a(x)y_2'' + b(x)y_2' + c(x)y_2 = 0.$$

Therefore,

$$a(x)y'' + b(x)y' + c(x)y = C_1(0) + C_2(0) = 0.$$

Hence,

$$y = C_1 y_1 + C_2 y_2$$

is also a solution of the homogeneous equation. ■

In the rest sections, we develop the theory and methods used to solve second-order linear ODEs and study their applications to real-world problems.

3 Existence and Uniqueness Theorem

Theorem — Existence and Uniqueness

Consider the IVP:

$$y'' + P(x)y' + Q(x)y = R(x), \qquad y(x_0) = y_0, \quad y'(x_0) = y_1$$

If $P(x)$, $Q(x)$, and $R(x)$ are **continuous** on an open interval I containing x_0, then there exists a **unique** solution $y = \phi(x)$ defined on the **entire** interval I.

> The interval of validity I is the largest interval containing x_0 where the coefficients remain continuous.

This theorem answers two important questions:

1. **Does a solution exist?**
2. **Is the solution unique?**

The theorem says:

- A solution **does exist**, and there is **only one** solution satisfying the given initial conditions, provided the coefficients are continuous.

Key points:

1. Why are two initial conditions needed?

A second-order differential equation contains the second derivative y'', so two integrations are usually required to obtain y.

Each integration introduces a constant:

$$C_1, \qquad C_2.$$

Therefore, two initial conditions are needed to determine these constants uniquely:

$$y(x_0) = y_0, \qquad y'(x_0) = y_1.$$

2. Importance of continuity

The theorem requires that

$$P(x), \quad Q(x), \quad R(x)$$

be continuous on the interval.

If one of them is discontinuous at some point, the theorem no longer guarantees existence and uniqueness beyond that point.

Example 1: Use the Existence and Uniqueness Theorem to determine the interval on which the IVP has a unique solution.

$$y'' + (\tan x)y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Here,

$$P(x) = \tan x, \qquad Q(x) = 1, \qquad R(x) = 0.$$

The function $\tan x$ is discontinuous at

$$x = \pm\frac{\pi}{2}.$$

Since the initial point is (x_0=0), the largest interval containing (0) where all coefficients are continuous is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, the theorem guarantees a unique solution on

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Example 2: Determine whether the following IVP has a unique solution near the initial point.

$$y'' + \frac{1}{x-2}y' + y = e^x, \qquad y(3) = 0, \qquad y'(3) = 1.$$

We identify

$$P(x) = \frac{1}{x-2}, \qquad Q(x) = 1, \qquad R(x) = e^x.$$

The function

$$\frac{1}{x-2}$$

is discontinuous at $x = 2$.

Therefore, by the Existence and Uniqueness Theorem, a unique solution exists on

$$(2, \infty).$$

Example 3 Find the largest interval containing the initial point for which the theorem guarantees a unique solution.

$$(x-4)y'' + y' + xy = \sin x, \qquad y(5) = 2, \qquad y'(5) = 0.$$

First write the equation in standard form by dividing by $x - 4$:

$$y'' + \frac{1}{x-4}y' + \frac{x}{x-4}y = \frac{\sin x}{x-4}.$$

Thus,

$$P(x) = \frac{1}{x-4}, \qquad Q(x) = \frac{x}{x-4}, \qquad R(x) = \frac{\sin x}{x-4}.$$

All are discontinuous at

$$x = 4.$$

Since the initial point is $x_0 = 5$, the largest interval containing 5 is $(4, \infty)$. Hence a unique solution is guaranteed on

$$(4, \infty).$$

TRY

Q1: Use the Existence and Uniqueness Theorem to determine the interval on which the IVP has a unique solution.

$$y'' + x^2y' + (\cos x)y = \ln(x^2 + 1),$$

with

$$y(1) = 0, \qquad y'(1) = 3.$$

Q2: Find the largest interval containing the initial point for which the theorem guarantees a unique solution.

$$y'' + \frac{1}{x-1}y' + y = e^x,$$

with

$$y(0) = 2, \qquad y'(0) = 1.$$

---

4 Linear Independence and the Wronskian

When solving a second-order homogeneous differential equation,

$$y'' + P(x)y' + Q(x)y = 0,$$

we need **two independent solutions** to form the general solution.

For example,

$$y = C_1y_1 + C_2y_2.$$

But this only works if y_1 and y_2 are **linearly independent**.

4.1 Linear Independence

> Definition
>
> Two functions are linearly dependent if one is a constant multiple of the other:
>
> $$y_2 = ky_1$$
>
> .
> for some constant k.
> Otherwise, they are called **linearly independent**.

Simple Idea

- **Dependent** → same type of function, only multiplied by a constant.
- **Independent** → genuinely different functions.

Examples:

- $y_1 = e^{2x}$, $y_2 = 3e^{2x}$: **Dependent** ($y_2 = 3y_1$).
- $y_1 = e^{2x}$, $y_2 = e^{-2x}$: **Independent** (neither is a multiple of the other).
- $y_1 = \sin x$, $y_2 = \cos x$: **Independent**.
- $y_1 = x^2$, $y_2 = x|x|$: Tricky — check using the Wronskian.

4.2 The Wronskian

> Definition — Wronskian
>
> The **Wronskian** of two differentiable functions y_1 and y_2 is:
>
> $$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Wronskian Test

- If $W(x) \neq 0$ for some x in the same interval → **independent**
- If $W(x) = 0$ everywhere → **dependent**

4.2.1 Examples using the Wronskian

Example 1: Show $y_1 = e^{2x}$ and $y_2 = e^{-3x}$ are linearly independent.

$$W = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -3e^{2x}e^{-3x} - 2e^{2x}e^{-3x} = -5e^{-x} \neq 0$$

Since $W \neq 0$ everywhere, they are **linearly independent**. ✓

Example 2: Are $y_1 = \sin 2x$ and $y_2 = \sin x \cos x$ linearly independent? Using the identity

$$\sin 2x = 2 \sin x \cos x,$$

we get

$$y_2 = \frac{1}{2} \sin 2x = \frac{1}{2} y_1.$$

Therefore, the functions are **linearly dependent**.

Note $y_2 = \frac{1}{2} \sin 2x = \frac{1}{2} y_1$. So $y_2 = \frac{1}{2} y_1$ — they are **linearly dependent**.

Check via Wronskian: $y_2' = \cos 2x$, $y_1' = 2 \cos 2x$.

$W = \sin 2x \cdot \cos 2x - \sin x \cos x \cdot 2 \cos 2x = \cos 2x(\sin 2x - 2 \sin x \cos x) = \cos 2x \cdot 0 = 0 \checkmark$

5 Fundamental Solutions

For a second-order homogeneous linear differential equation,

$$y'' + P(x)y' + Q(x)y = 0,$$

we need two linearly independent solutions to form the general solution.

These special solutions are called a **fundamental set of solutions**.

> Definition 3.3 — Fundamental Set of Solutions
>
> A pair $\{y_1, y_2\}$ of **linearly independent** solutions of the homogeneous equation on I is called a **fundamental set of solutions**.
> The **general solution** of the homogeneous equation is:
>
> $$y_h = C_1 y_1(x) + C_2 y_2(x), \quad C_1, C_2 \in \mathbb{R} (\text{are arbitrary constants})$$

Why do we need two solutions?

A second-order differential equation contains the second derivative y''. Solving such an equation introduces two arbitrary constants.

Therefore, the complete solution must contain two independent functions.

If the solutions are not independent, they cannot produce the full family of solutions.

5.1 Existence of Fundamental Solutions

The existence and uniqueness theorem guarantees that a fundamental set always exists.

We can obtain two independent solutions by choosing different initial conditions.

Example

Consider

$$y'' + y = 0.$$

Choose a first solution y_1 satisfying

$$y_1(0) = 1, \qquad y_1'(0) = 0.$$

This gives

$$y_1 = \cos x.$$

Now choose a second solution y_2 satisfying

$$y_2(0) = 0, \qquad y_2'(0) = 1.$$

This gives

$$y_2 = \sin x.$$

Check Linear Independence

Compute the Wronskian:

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}.$$

Thus,

$$W = \cos^2 x + \sin^2 x = 1.$$

Since

$$W \neq 0,$$

the functions are linearly independent.

Therefore,

$$\{\cos x, \sin x\}$$

forms a fundamental set of solutions:

$$y = C_1 \cos x + C_2 \sin x.$$

5.2 Linear Equations with Constant Coefficients

We now study the most important special case:

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants.

For these equations, there is a systematic method for finding a fundamental set of solutions.

5.3 The Characteristic Equation

To solve the equation, we try an exponential solution of the form

$$y = e^{rx},$$

where r is a constant to be determined. Why do we choose an exponential function?

We choose the exponential function because its derivatives are very simple, such that

Then

$$y' = re^{rx}, \qquad y'' = r^2 e^{rx}.$$

Notice something important:

> Every derivative is still a multiple of the same function e^{rx}.

This makes substitution into the differential equation very easy.

Substitute y' and y'' into

$$ay'' + by' + cy = 0 :$$

$$a(r^2 e^{rx}) + b(re^{rx}) + c(e^{rx}) = 0.$$

Factor out e^{rx}:

$$e^{rx}(ar^2 + br + c) = 0.$$

Since

$$e^{rx} \neq 0,$$

we obtain

$$ar^2 + br + c = 0.$$

This is called the **characteristic equation** (or auxiliary equation).

So the differential equation becomes an algebraic equation.

> This is why exponential functions are extremely useful here.
>
> ---
>
> **Why not try polynomials or trigonometric functions first?**
> Suppose we try
>
> $$y = x^2.$$
>
> Then
>
> $$y' = 2x, \qquad y'' = 2.$$
>
> The derivatives change form completely.
> There is no common factor to simplify the equation nicely.
>
> ---
>
> Now suppose we try
>
> $$y = \sin x.$$
>
> Then
>
> $$y' = \cos x, \qquad y'' = -\sin x.$$
>
> The derivatives alternate between sine and cosine.
> This is less convenient than exponentials.
>
> ---
>
> **Important Observation**
> Exponential functions are special because:
>
> $$\text{Derivative of } e^{rx} = \text{constant} \times e^{rx}.$$

This property perfectly matches linear differential equations with constant coefficients.
That is why the exponential trial works naturally.

The roots of this equation determine the form of the fundamental solutions.

The discriminant

$$\Delta = b^2 - 4ac$$

determines the type of roots. We have three cases:

1. Two distinct real roots
2. Repeated real root
3. Complex conjugate roots

5.3.1 Case 1: Two Distinct Real Roots ($\Delta > 0$)

Suppose the characteristic equation has two distinct real roots

$$r_1 \neq r_2.$$

Then the two independent solutions are

$$e^{r_1 x} \quad \text{and} \quad e^{r_2 x}.$$

Hence the **General solution** is

$$\boxed{y = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

Example 1: Solve $y'' - 5y' + 6y = 0$.

Characteristic equation: $r^2 - 5r + 6 = 0$

$$(r-2)(r-3) = 0 \quad \Rightarrow \quad r_1 = 2,\ r_2 = 3$$

$$\boxed{y = C_1 e^{2x} + C_2 e^{3x}}$$

Example 2: Solve $2y'' + y' - y = 0$.

$2r^2 + r - 1 = 0 \Rightarrow (2r-1)(r+1) = 0 \Rightarrow r_1 = \dfrac{1}{2},\ r_2 = -1$

$$\boxed{y = C_1 e^{x/2} + C_2 e^{-x}}$$

Example 3

Solve

$$y'' - y = 0.$$

Step 1: Form the characteristic equation

Assume a solution of the form

$$y = e^{rx}.$$

Then

$$y' = re^{rx}, \qquad y'' = r^2 e^{rx}.$$

Substitute into the differential equation:

$$r^2 e^{rx} - e^{rx} = 0.$$

Factor out $e^{rx} \neq 0$:

$$e^{rx}(r^2 - 1) = 0.$$

Hence the characteristic equation is

$$r^2 - 1 = 0.$$

Step 2: Solve the characteristic equation

Factor:

$$(r - 1)(r + 1) = 0.$$

Therefore,

$$r_1 = 1, \qquad r_2 = -1.$$

These are two distinct real roots.

Step 3: Write the general solution

For distinct real roots,

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Substitute the roots:

$$\boxed{y = C_1 e^{x} + C_2 e^{-x}}$$

5.3.1.1 Hyperbolic Functions

The solution can also be written using **hyperbolic functions**.

These are functions closely related to exponentials.

Definition of Hyperbolic Cosine

The hyperbolic cosine is defined by

$$\cosh x = \frac{e^{x} + e^{-x}}{2}.$$

Definition of Hyperbolic Sine

The hyperbolic sine is defined by

$$\sinh x = \frac{e^{x} - e^{-x}}{2}.$$

5.3.1.2 Why are they useful?

Notice:

- $\cosh x$ combines e^{x} and e^{-x},
- $\sinh x$ also combines e^{x} and e^{-x}.

Since our solution already contains

$$e^{x} \quad \text{and} \quad e^{-x},$$

it can be rewritten using $\cosh x$ and $\sinh x$.

Rewriting the Solution

Start with

$$y = C_1 e^x + C_2 e^{-x}.$$

Using the definitions:

$$e^x = \cosh x + \sinh x,$$

and

$$e^{-x} = \cosh x - \sinh x.$$

Substitute into the solution:

$$y = C_1(\cosh x + \sinh x) + C_2(\cosh x - \sinh x).$$

Expand:

$$y = (C_1 + C_2)\cosh x + (C_1 - C_2)\sinh x.$$

Rename the constants:

$$A = C_1 + C_2, \qquad B = C_1 - C_2.$$

Then

$$\boxed{y = A\cosh x + B\sinh x}$$

which is equivalent to

$$\boxed{y = C_1 e^x + C_2 e^{-x}.}$$

Important Note

Both forms represent exactly the same family of solutions.

You may use either:

$$y = C_1 e^x + C_2 e^{-x}$$

or

$$y = A \cosh x + B \sinh x.$$

The exponential form is more common in differential equations, while the hyperbolic form is common in physics and engineering.

5.3.2 Case 2: Repeated Real Root ($\Delta = 0$)

Single root $r_1 = r_2 = r = -\dfrac{b}{2a}$. A second independent solution is xe^{rx}.

General solution:

$$\boxed{y = (C_1 + C_2 x)e^{rx}}$$

Why xe^{rx}? The method of reduction of order: knowing $y_1 = e^{rx}$, set $y_2 = v(x)e^{rx}$. Substituting into the ODE eventually gives $v'' = 0$, so $v = C_1 + C_2 x$. Taking $v = x$ gives the second independent solution.

Example 1: Solve $y'' - 6y' + 9y = 0$.

$r^2 - 6r + 9 = 0 \Rightarrow (r - 3)^2 = 0 \Rightarrow r = 3$ (repeated)

$$\boxed{y = (C_1 + C_2 x)e^{3x}}$$

Example 2: Solve $y'' + 4y' + 4y = 0$.

$r^2 + 4r + 4 = 0 \Rightarrow (r + 2)^2 = 0 \Rightarrow r = -2$ (repeated)

$$\boxed{y = (C_1 + C_2 x)e^{-2x}}$$

Example 3: Solve $4y'' - 4y' + y = 0$.

$4r^2 - 4r + 1 = 0 \Rightarrow (2r - 1)^2 = 0 \Rightarrow r = \dfrac{1}{2}$

$$\boxed{y = (C_1 + C_2 x)\, e^{x/2}}$$

5.3.3 Case 3: Complex Conjugate Roots ($\Delta < 0$)

When the characteristic equation has negative discriminant,

$$\Delta = b^2 - 4ac < 0,$$

the roots are complex numbers.

They occur in the form

$$r = \alpha \pm \beta i,$$

where

$$\alpha = -\frac{b}{2a}, \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

5.3.3.1 Why do sine and cosine appear?

Suppose one root is

$$r = \alpha + \beta i.$$

Then one exponential solution is

$$y = e^{(\alpha+\beta i)x}.$$

Using exponent laws:

$$e^{(\alpha+\beta i)x} = e^{\alpha x} e^{i\beta x}.$$

Now apply Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Replacing θ by βx:

$$e^{i\beta x} = \cos\beta x + i\sin\beta x.$$

Therefore,

$$e^{(\alpha+\beta i)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x).$$

Similarly, the second root gives

$$e^{(\alpha-\beta i)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

From these two complex solutions, we obtain two real independent solutions:

$$e^{\alpha x}\cos\beta x \quad \text{and} \quad e^{\alpha x}\sin\beta x.$$

Hence the general solution becomes

$$\boxed{y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)}$$

Example 1

Solve

$$y'' + 4y = 0.$$

Step 1: Form the characteristic equation

Assume

$$y = e^{rx}.$$

Then

$$y' = re^{rx}, \qquad y'' = r^2e^{rx}.$$

Substitute into the equation:

$$r^2e^{rx} + 4e^{rx} = 0.$$

Factor out $e^{rx} \neq 0$:

$$e^{rx}(r^2 + 4) = 0.$$

Hence the characteristic equation is

$$r^2 + 4 = 0.$$

Step 2: Solve for the roots

$$r^2 = -4.$$

Take square roots:

$$r = \pm 2i.$$

Thus,

$$\alpha = 0, \qquad \beta = 2.$$

Step 3: Write the complex exponential solutions

The roots give

$$e^{2ix} \quad \text{and} \quad e^{-2ix}.$$

Using Euler's formula:

$$e^{2ix} = \cos 2x + i \sin 2x.$$

Thus the real independent solutions are

$$\cos 2x \quad \text{and} \quad \sin 2x.$$

Step 4: Write the general solution

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x}$$

Example 2

Solve

$$y'' - 2y' + 5y = 0.$$

Step 1: Form the characteristic equation

$$r^2 - 2r + 5 = 0.$$

Step 2: Solve using the quadratic formula

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}.$$

Simplify:

$$r = \frac{2 \pm \sqrt{4 - 20}}{2}.$$

$$= \frac{2 \pm \sqrt{-16}}{2}.$$

Since

$$\sqrt{-16} = 4i,$$

we get

$$r = \frac{2 \pm 4i}{2}.$$

Therefore,

$$r = 1 \pm 2i.$$

Thus,

$$\alpha = 1, \qquad \beta = 2.$$

Step 3: Form the general solution

Using

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x),$$

substitute $\alpha = 1$ and $\beta = 2$:

$$\boxed{y = e^{x}(C_1 \cos 2x + C_2 \sin 2x)}$$

5.3.3.2 Interpretation

Notice the solution has two parts:

Oscillatory part

$$\cos 2x, \qquad \sin 2x$$

These produce oscillation.

Exponential part

$$e^x$$

This controls growth.

So the oscillation grows larger as x increases.

Example 3

Solve

$$y'' + 2y' + 10y = 0.$$

Step 1: Characteristic equation

$$r^2 + 2r + 10 = 0.$$

Step 2: Solve using the quadratic formula

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2}.$$

Simplify:

$$r = \frac{-2 \pm \sqrt{4 - 40}}{2}.$$

$$= \frac{-2 \pm \sqrt{-36}}{2}.$$

Since

$$\sqrt{-36} = 6i,$$

we obtain

$$r = \frac{-2 \pm 6i}{2}.$$

Thus,

$$r = -1 \pm 3i.$$

Hence,

$$\alpha = -1, \qquad \beta = 3.$$

Step 3: Write the general solution

$$\boxed{y = e^{-x}(C_1 \cos 3x + C_2 \sin 3x)}$$

5.3.3.3 Interpretation

Again, we have:

Oscillation

$$\cos 3x, \qquad \sin 3x$$

Exponential factor

$$e^{-x}$$

Since e^{-x} decreases as $x \to \infty$,

the oscillations gradually shrink.

This is called a **damped oscillation**.

5.4 Summary of Complex Roots

If the characteristic roots are

$$r = \alpha \pm \beta i,$$

then the general solution is

$$\boxed{y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)}$$

where:

- $e^{\alpha x}$ controls growth or decay,
- $\cos \beta x$ and $\sin \beta x$ produce oscillation,
- β determines the frequency of oscillation.

5.5 Summary Table — Three Cases

Discriminant	Roots	General Solution
$\Delta = b^2 - 4ac > 0$	$r_1 \neq r_2$ (real)	$C_1 e^{r_1 x} + C_2 e^{r_2 x}$
$\Delta = 0$	$r_1 = r_2 = r$ (repeated)	$(C_1 + C_2 x)e^{rx}$
$\Delta < 0$	$r = \alpha \pm \beta i$ (complex)	$e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

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6 Non-Homogeneous Equations: Structure of the General Solution

For the non-homogeneous equation $ay'' + by' + cy = g(x)$:

Theorem 3.4 — General Solution Structure

$$y_{\text{general}} = y_h + y_p$$

where:

- $y_h = C_1 y_1 + C_2 y_2$ is the **complementary function** (general solution of the associated homogeneous equation)
- y_p is **any particular solution** of the non-homogeneous equation

Why? If y_p satisfies $ay_p'' + by_p' + cy_p = g(x)$, and y_h satisfies $ay_h'' + by_h' + cy_h = 0$, then $y = y_h + y_p$ satisfies the full equation.

We study two methods for finding y_p: **undetermined coefficients** and **variation of parameters**.

6.1 Method of Undetermined Coefficients

This method works when $g(x)$ is a **polynomial**, **exponential**, **sine/cosine**, or **products** of these.

Principle: The form of y_p "mirrors" the form of $g(x)$.

6.1.1 Trial Function Table

Form of $g(x)$	Trial y_p
k (constant)	A
kx^n	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0$
ke^{ax}	Ae^{ax}
$k \sin(bx)$	$A \sin(bx) + B \cos(bx)$
$k \cos(bx)$	$A \sin(bx) + B \cos(bx)$
$ke^{ax} \sin(bx)$	$e^{ax}(A \sin bx + B \cos bx)$
$ke^{ax} x^n$	$e^{ax}(A_n x^n + \cdots + A_0)$
Sum of above	Sum of corresponding trials

> ⚠ Modification Rule
>
> If **any term** of the trial y_p duplicates a term already present in y_h, multiply the **entire** trial function by x. If duplication persists (repeated root), multiply by x^2.

Example 1: Solve $y'' - 3y' + 2y = 5e^{3x}$.

Step 1 — Homogeneous solution: $r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0 \Rightarrow r = 1, 2$

$y_h = C_1 e^x + C_2 e^{2x}$

Step 2 — Trial: $g = 5e^{3x}$. Try $y_p = Ae^{3x}$.

$y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$.

Step 3 — Substitute:

$$9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 5e^{3x} \Rightarrow 2A = 5 \Rightarrow A = \frac{5}{2}$$

$$\boxed{y = C_1 e^x + C_2 e^{2x} + \frac{5}{2} e^{3x}}$$

Example 2: Solve $y'' + y = 3\cos 2x$.

Step 1: $r^2 + 1 = 0 \Rightarrow r = \pm i$. $y_h = C_1 \cos x + C_2 \sin x$.

Step 2: $g = 3\cos 2x$. Trial: $y_p = A\cos 2x + B\sin 2x$.

Step 3: $y_p'' = -4A\cos 2x - 4B\sin 2x$.

$$(-4A + A)\cos 2x + (-4B + B)\sin 2x = 3\cos 2x$$
$$-3A\cos 2x - 3B\sin 2x = 3\cos 2x$$

Comparing: $-3A = 3 \Rightarrow A = -1$; $-3B = 0 \Rightarrow B = 0$.

$$\boxed{y = C_1 \cos x + C_2 \sin x - \cos 2x}$$

Example 3: Solve $y'' - y' = 2x$.

Step 1: $r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0, 1$.

$y_h = C_1 + C_2 e^x$.

Step 2: $g = 2x$ (polynomial degree 1). Trial: $y_p = Ax + B$.

But B (constant) is a solution of the homogeneous (since $r = 0$). **Modification:** Multiply by x:

$y_p = x(Ax + B) = Ax^2 + Bx$.

Step 3: $y_p' = 2Ax + B$, $y_p'' = 2A$.

$$2A - (2Ax + B) = 2x$$
$$-2Ax + (2A - B) = 2x$$

Comparing: $-2A = 2 \Rightarrow A = -1$; $2A - B = 0 \Rightarrow B = -2$.

$$\boxed{y = C_1 + C_2 e^x - x^2 - 2x}$$

Example 4 (Modification Rule — resonance): Solve $y'' + 4y = 3\sin 2x$.

Step 1: $r^2 + 4 = 0 \Rightarrow r = \pm 2i$. $y_h = C_1\cos 2x + C_2\sin 2x$.

Step 2: Trial $A\cos 2x + B\sin 2x$ — this **duplicates** y_h! Multiply by x:

$y_p = x(A\cos 2x + B\sin 2x) = Ax\cos 2x + Bx\sin 2x$.

Step 3: Compute y_p'' (product rule):

$y_p' = A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x$

$y_p'' = -4A\sin 2x - 4Ax\cos 2x + 4B\cos 2x - 4Bx\sin 2x$

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x = 3\sin 2x$$

Comparing: $-4A = 3 \Rightarrow A = -\dfrac{3}{4}$; $4B = 0 \Rightarrow B = 0$.

$$\boxed{y = C_1\cos 2x + C_2\sin 2x - \frac{3}{4}x\cos 2x}$$

> The term $-\frac{3}{4}x\cos 2x$ grows without bound — this is **resonance**, the physical phenomenon where a driving frequency matches the natural frequency.

Example 5: Solve $y'' + 3y' + 2y = e^{-x}(1 + 2x)$.

Step 1: $r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0 \Rightarrow r = -1, -2$.

$y_h = C_1e^{-x} + C_2e^{-2x}$.

Step 2: $g = e^{-x} + 2xe^{-x}$. Trial: $e^{-x}(Ax + B)$.

But e^{-x} is in y_h (root $r = -1$). Multiply by x:

$y_p = xe^{-x}(Ax + B) = (Ax^2 + Bx)e^{-x}$.

Step 3: Compute derivatives (using product rule):

$y_p' = (2Ax + B)e^{-x} - (Ax^2 + Bx)e^{-x} = e^{-x}(-Ax^2 + (2A - B)x + B)$

$y_p'' = e^{-x}(Ax^2 + (-4A + B)x + (2A - 2B))$

Substituting into the ODE and collecting e^{-x} terms:

Coefficient of x^2: $A - 3A + 2A = 0$

Coefficient of x: $(-4A + B) + 3(2A - B) + 2B = -4A + B + 6A - 3B + 2B = 2A = 2 \Rightarrow A = 1$

Constant: $(2A - 2B) + 3B + 0 = 2A + B = 2 + B = 1 \Rightarrow B = -1$

$$\boxed{y = C_1 e^{-x} + C_2 e^{-2x} + (x^2 - x)e^{-x}}$$

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7 Appendices: Exponential Forms of Trigonometric and Hyperbolic Functions

Exponential functions are deeply connected to trigonometric and hyperbolic functions.

These relationships are very important in differential equations, complex analysis, physics, and engineering.

7.1 Euler's Formula

The connection begins with the famous Euler formula:

$$e^{ix} = \cos x + i \sin x$$

Replacing x by $-x$:

$$e^{-ix} = \cos x - i \sin x$$

These two formulas allow us to express trigonometric functions using exponentials.

7.2 Trigonometric Functions in Exponential Form

7.2.1 Cosine

Add the two Euler formulas:

$$e^{ix} + e^{-ix} = (\cos x + i \sin x) + (\cos x - i \sin x).$$

The imaginary parts cancel:

$$e^{ix} + e^{-ix} = 2 \cos x.$$

Hence,

$$\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}}$$

7.2.2 Sine

Subtract the Euler formulas:

$$e^{ix} - e^{-ix} = (\cos x + i \sin x) - (\cos x - i \sin x).$$

Simplify:

$$e^{ix} - e^{-ix} = 2i \sin x.$$

Therefore,

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

7.2.3 Tangent

Since

$$\tan x = \frac{\sin x}{\cos x},$$

we obtain

$$\boxed{\tan x = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}}$$

7.3 Hyperbolic Functions

Hyperbolic functions are built directly from real exponentials.

7.3.1 Hyperbolic Cosine

$$\boxed{\cosh x = \frac{e^{x} + e^{-x}}{2}}$$

7.3.2 Hyperbolic Sine

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$$

7.3.3 Hyperbolic Tangent

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

Thus,

$$\boxed{\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}}$$

7.4 Other Hyperbolic Functions

7.4.1 Hyperbolic Secant

$$\boxed{\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}}$$

7.4.2 Hyperbolic Cosecant

$$\boxed{\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}}$$

7.4.3 Hyperbolic Cotangent

$$\boxed{\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}}$$

7.5 Comparison Between Trigonometric and Hyperbolic Functions

Trigonometric	Exponential Form
$\cos x$	$\frac{e^{ix} + e^{-ix}}{2}$
$\sin x$	$\frac{e^{ix} - e^{-ix}}{2i}$

Trigonometric	Exponential Form
$\tan x$	$\dfrac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$

Hyperbolic	Exponential Form
$\cosh x$	$\dfrac{e^x + e^{-x}}{2}$
$\sinh x$	$\dfrac{e^x - e^{-x}}{2}$
$\tanh x$	$\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

7.6 Important Observation

Notice the similarity:

7.6.1 Trigonometric functions

Use **complex exponentials**:

$$e^{ix}, \qquad e^{-ix}.$$

7.6.2 Hyperbolic functions

Use **real exponentials**:

$$e^{x}, \qquad e^{-x}.$$

This is why:

- complex roots in differential equations produce sine and cosine;
- real roots involving e^x and e^{-x} naturally produce hyperbolic functions.

7.7 Useful Identities

7.7.1 Trigonometric Identity

$$\boxed{\cos^2 x + \sin^2 x = 1}$$

7.7.2 Hyperbolic Identity

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

Notice the sign difference:

- trigonometric identity uses $+$,
- hyperbolic identity uses $-$.
