MAT 102: Ordinary Differential Equations

Topic 4 — Laplace Transforms and Applications to Differential Equations

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2026-06-30

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What You Will Learn in This Topic

By the end of this topic, you should be able to:

1. Explain what an improper integral is and evaluate basic ones
2. Define the Laplace transform and compute it from the definition
3. Use standard transform tables and theorems to find transforms quickly
4. Find inverse Laplace transforms using partial fractions
5. Apply Laplace transforms step-by-step to solve ODEs with initial conditions
6. Solve systems of simultaneous differential equations using Laplace transforms

1 Background Ideas

Before we can understand the Laplace transform, we need to be comfortable with two things used directly in its definition: **improper integrals** and **integration by parts**.

1.1 What Is an Improper Integral?

In ordinary integration we compute $\int_a^b f(t)\,dt$ where both limits are **finite**. An **improper integral** is one where one or both limits is **infinity**, for example:

$$\int_0^\infty e^{-2t}\,dt$$

Since ∞ is not a real number we cannot substitute it directly. Instead we use a limit.

Definition: Improper Integral with Upper Limit ∞

$$\int_0^\infty f(t)\,dt \;=\; \lim_{T\to\infty}\int_0^T f(t)\,dt$$

- If the limit is a **finite number**, the integral **converges** to that number.
- If the limit is $\pm\infty$ or does not exist, the integral **diverges**.

Example 1: Evaluate $\int_0^\infty e^{-2t}\,dt$.

Step 1 — Replace ∞ with T and integrate:

$$\int_0^T e^{-2t}\,dt = \left[\frac{e^{-2t}}{-2}\right]_0^T = -\frac{e^{-2T}}{2} + \frac{1}{2}$$

Step 2 — Take the limit (since $s > 0$, $e^{-2T} \to 0$ as $T \to \infty$):

$$\lim_{T\to\infty}\left(-\frac{e^{-2T}}{2} + \frac{1}{2}\right) = 0 + \frac{1}{2}$$

$$\boxed{\int_0^{\infty} e^{-2t}\,dt = \frac{1}{2}}$$

Example 2: Evaluate $\displaystyle\int_0^{\infty} e^{3t}\,dt$.

$$\int_0^{T} e^{3t}\,dt = \frac{e^{3T}}{3} - \frac{1}{3} \xrightarrow{T\to\infty} \infty$$

$$\boxed{\int_0^{\infty} e^{3t}\,dt \text{ diverges}}$$

Tip

Key pattern:

- Negative exponent e^{-at}, $a > 0 \;\Rightarrow\;$ shrinks to $0 \;\Rightarrow\;$ integral **converges**
- Positive exponent e^{at}, $a > 0 \;\Rightarrow\;$ grows without bound $\;\Rightarrow\;$ integral **diverges**

Example 3: Evaluate $\displaystyle\int_0^{\infty} e^{-st}\,dt$ for constant $s > 0$.

$$\int_0^{T} e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^{T} = -\frac{e^{-sT}}{s} + \frac{1}{s} \xrightarrow{T\to\infty} \frac{1}{s}$$

$$\boxed{\int_0^{\infty} e^{-st}\,dt = \frac{1}{s}, \qquad s > 0}$$

We use this result repeatedly throughout the topic.

1.2 Integration by Parts — A Quick Refresher

$$\int u\,dv = uv - \int v\,du$$

Use the **LIATE rule** to choose u:

Priority	Type	Examples
1st	**L**ogarithms	$\ln t$
2nd	**I**nverse trig	$\arctan t$
3rd	**A**lgebraic	$t,\ t^2$
4th	**T**rigonometric	$\sin t,\ \cos t$
5th	**E**xponentials	e^{-2t}

Example: Evaluate $\displaystyle\int t\,e^{-st}\,dt$.

Choose $u = t$, $dv = e^{-st}\,dt$, so $du = dt$, $v = \dfrac{e^{-st}}{-s}$:

$$\int t\,e^{-st}\,dt = \frac{-t\,e^{-st}}{s} - \int \frac{e^{-st}}{-s}\,dt = \frac{-t\,e^{-st}}{s} - \frac{e^{-st}}{s^2} + C$$

We use this result when computing $\mathcal{L}\{t\}$.

2 Introduction — Why Do We Need Laplace Transforms?

2.1 The Problem with Our Current Methods

In Topics 2 and 3 we learned integrating factors, undetermined coefficients, and variation of parameters. These methods solve the equation first and apply initial conditions at the end, which can be long and error-prone for complicated forcing functions.

The Laplace Transform Strategy

$$\text{ODE (in } t) \xrightarrow{\text{Step 1: Transform}} \text{Algebraic equation (in } s)$$

$$\xrightarrow{\text{Step 2: Solve for } Y(s)} \text{Answer in } s\text{-world}$$

$$\xrightarrow{\text{Step 3: Invert}} \text{Solution } y(t)$$

Why is this better?

- Derivatives become simple algebra: y' becomes $sY - y(0)$
- Initial conditions are included automatically — not added at the end
- Complicated forcing functions are handled naturally

Analogy: To multiply $1{,}234 \times 5{,}678$ you could take logarithms, add, then convert back. Same idea — transform to a simpler world, work there, convert back.

3 Definition of the Laplace Transform

3.1 Building the Definition Step by Step

We multiply $f(t)$ by e^{-st} and integrate over $[0, \infty)$. The factor e^{-st} is a **damping weight**: for $s > 0$ it forces $e^{-st} f(t) \to 0$ as $t \to \infty$, ensuring convergence for a wide class of functions.

Definition 4.1 — The Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The integral must converge for the transform to exist.

Key Remarks

1. t disappears after integration — the result is a function of s only.
2. s is a new variable (the "frequency domain"). Think of it as a parameter.
3. **Capital letters for transforms:** $\mathcal{L}\{y(t)\} = Y(s)$, $\mathcal{L}\{x(t)\} = X(s)$, etc.
4. s must be large enough for the integral to converge.

3.2 Computing the Transform from the Definition — Four Core Examples

Example 1: Find $\mathcal{L}\{1\}$.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s} \quad \text{(from Example 3 above)}$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0}$$

Example 2: Find $\mathcal{L}\{e^{at}\}$.

$$\int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a} \quad (s - a > 0)$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad s > a}$$

$f(t)$	a	$F(s)$	Valid when
e^{3t}	3	$\frac{1}{s-3}$	$s > 3$
e^{-5t}	-5	$\frac{1}{s+5}$	$s > -5$
e^{-t}	-1	$\frac{1}{s+1}$	$s > -1$

Example 3: Find $\mathcal{L}\{t\}$.

Using the integration by parts result from Section 1.2:

$$\mathcal{L}\{t\} = \lim_{T\to\infty} \left[\frac{-t\,e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^T$$

At the upper limit both terms $\to 0$ (exponential decay beats linear growth). At $t = 0$ both terms give $0 - (-1/s^2) = 1/s^2$:

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}, \qquad s > 0}$$

General pattern:

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad s > 0}$$

Examples: $\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad \mathcal{L}\{t^3\} = \frac{6}{s^4}, \quad \mathcal{L}\{t^5\} = \frac{120}{s^6}.$

Example 4: Prove that evaluating the improper integral limits of the antiderivative yields the standard Laplace transform formula:

$$\boxed{\mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}, \qquad s > 0}$$

Step-by-Step Derivation

1. First Integration by Parts Following the LIATE rule, Trigonometric (**T**) functions take priority over Exponential (**E**) functions when choosing u.

$$\begin{aligned} u &= \sin(bt) \qquad & dv &= e^{-st}dt \\ du &= b\cos(bt)dt \qquad & v &= -\frac{1}{s}e^{-st} \end{aligned}$$

Applying the integration by parts formula $\int u\,dv = uv - \int v\,du$:

$$\begin{aligned} I &= -\frac{1}{s}e^{-st}\sin bt - \int \left(-\frac{1}{s}e^{-st}\right)(b\cos bt)\,dt \\ I &= -\frac{1}{s}e^{-st}\sin bt + \frac{b}{s}\int e^{-st}\cos bt\,dt \end{aligned} \tag{1}$$

Second Integration by Parts We apply integration by parts a second time to the remaining integral $\int e^{-st}\cos bt\,dt$, keeping the same assignment priorities.

$$
\begin{aligned}
u &= \cos(bt) \qquad & dv &= e^{-st} dt \\
du &= -b\sin(bt)dt \qquad & v &= -\frac{1}{s}e^{-st}
\end{aligned}
$$

Substitute this back into the new integral component:

$$
\int e^{-st}\cos bt\,dt = -\frac{1}{s}e^{-st}\cos bt - \int \left(-\frac{1}{s}e^{-st}\right)(-b\sin bt)\,dt
$$

$$
\int e^{-st}\cos bt\,dt = -\frac{1}{s}e^{-st}\cos bt - \frac{b}{s}\int e^{-st}\sin bt\,dt
$$

Notice that our original integral I in (Equation 1) has reappeared inside this expression:

$$
\int e^{-st}\cos bt\,dt = -\frac{1}{s}e^{-st}\cos bt - \frac{b}{s}I \tag{2}
$$

Algebraic Isolation Substitute the result from Equation 2 into back into the Equation 1:

$$
I = -\frac{1}{s}e^{-st}\sin bt + \frac{b}{s}\left(-\frac{1}{s}e^{-st}\cos bt - \frac{b}{s}I\right)
$$

$$
I = -\frac{1}{s}e^{-st}\sin bt - \frac{b}{s^2}e^{-st}\cos bt - \frac{b^2}{s^2}I
$$

Group all terms containing I on the left side of the equation:

$$
I + \frac{b^2}{s^2}I = e^{-st}\left(-\frac{1}{s}\sin bt - \frac{b}{s^2}\cos bt\right)
$$

Find a common denominator of s^2 for both sides to easily combine fractions:

$$
\left(\frac{s^2+b^2}{s^2}\right)I = e^{-st}\left(\frac{-s\sin bt - b\cos bt}{s^2}\right)
$$

Multiply both sides by s^2 to eliminate the denominators:

$$(s^2 + b^2)I = e^{-st}(-s \sin bt - b \cos bt)$$

Finally, divide through by $(s^2 + b^2)$ to isolate I:

$$I = \frac{e^{-st}(-s \sin bt - b \cos bt)}{s^2 + b^2}$$

> **Note**
>
> Adding an integration constant $+C$ completes the indefinite integral form.

$$\int e^{-st} \sin bt \, dt = \frac{e^{-st}(-s \sin bt - b \cos bt)}{s^2 + b^2} + C$$

$$\boxed{\mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}, \qquad s > 0}$$

Given the antiderivative:

$$\int e^{-st} \sin bt \, dt = \frac{e^{-st}(-s \sin bt - b \cos bt)}{s^2 + b^2}$$

Set Up the Improper Integral By definition, the Laplace transform is an improper integral running from 0 to ∞. We rewrite this using a formal limit with an upper bound $T \to \infty$:

$$\mathcal{L}\{\sin bt\} = \int_0^\infty e^{-st} \sin bt \, dt = \lim_{T \to \infty} \left[\frac{e^{-st}(-s \sin bt - b \cos bt)}{s^2 + b^2} \right]_0^T$$

Evaluate the Upper Limit ($t \to \infty$) We substitute T into the antiderivative and take the limit as $T \to \infty$:

$$\lim_{T \to \infty} \frac{e^{-sT}(-s \sin bT - b \cos bT)}{s^2 + b^2}$$

- **Exponential Behavior:** Under the constraint $s > 0$, the exponent $-sT$ remains strictly negative. As $T \to \infty$, the exponential term decays to zero: $\lim_{T\to\infty} e^{-sT} = 0$.
- **Trigonometric Behavior:** The terms $\sin bT$ and $\cos bT$ oscillate endlessly but are strictly bounded between -1 and 1.
- **Product Rule for Limits:** A decaying exponential term multiplied by a bounded oscillating expression vanishes completely at infinity.

$$\text{Upper Limit Value} = 0$$

Evaluate the Lower Limit ($t = 0$) Next, we substitute $t = 0$ into the expression:

$$\text{Lower Limit Value} = \frac{e^{-s(0)}(-s\sin(b\cdot 0) - b\cos(b\cdot 0))}{s^2+b^2}$$

Evaluating the elemental function values at 0: * $e^0 = 1$ * $\sin(0) = 0$ * $\cos(0) = 1$

Substitute these values back into the fraction to simplify:

$$\text{Lower Limit Value} = \frac{1\cdot(-s(0) - b(1))}{s^2+b^2} = \frac{-b}{s^2+b^2}$$

Apply the Fundamental Theorem of Calculus Subtract the lower limit value from the upper limit value (Upper − Lower):

$$\mathcal{L}\{\sin bt\} = 0 - \left(\frac{-b}{s^2+b^2}\right)$$

Simplifying the double negative yields the final proof:

$$\boxed{\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \qquad s > 0}$$

Your Turn: Prove

$$\boxed{\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}, \qquad s > 0}$$

Note: $\sin bt$ has numerator b; $\cos bt$ has numerator s.

$f(t)$	b	$F(s)$
$\sin 3t$	3	$\frac{3}{s^2+9}$
$\cos 3t$	3	$\frac{s}{s^2+9}$
$\sin 7t$	7	$\frac{7}{s^2+49}$
$\cos t$	1	$\frac{s}{s^2+1}$

4 Standard Transforms Reference Table

#	Function / Property Name	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	Condition
1	Unit Step / Constant	1	$\frac{1}{s}$	$s > 0$
2	Ramp Function	t	$\frac{1}{s^2}$	$s > 0$
3	Power Function $(n = 1, 2, 3, \ldots)$	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
4	Exponential Decay / Growth	e^{at}	$\frac{1}{s-a}$	$s > a$
5	Sine Wave	$\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0$
6	Cosine Wave	$\cos bt$	$\frac{s}{s^2+b^2}$	$s > 0$

#	Function / Property Name	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	Condition
7	Hyperbolic Sine	$\sinh bt$	$\frac{b}{s^2 - b^2}$	$s > \|b\|$
8	Hyperbolic Cosine	$\cosh bt$	$\frac{s}{s^2 - b^2}$	$s > \|b\|$
9	Exponentially Scaled Power	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
10	Damped Sine Wave	$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
11	Damped Cosine Wave	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
12	Heaviside Step Function	$u(t-a)$	$\frac{e^{-as}}{s}$	$s > 0$ $(a \geq 0)$
13	Dirac Delta (Unit Impulse)	$\delta(t-a)$	e^{-as}	— $(a \geq 0)$
14	Time-Shifted Function	$u(t-a)f(t-a)$	$e^{-as}F(s)$	—
15	Resonant Sine Combination	$t \sin bt$	$\frac{2bs}{(s^2 + b^2)^2}$	$s > 0$
16	Resonant Cosine Combination	$t \cos bt$	$\frac{s^2 - b^2}{(s^2 + b^2)^2}$	$s > 0$
17	Damped Hyperbolic Sine	$e^{at} \sinh bt$	$\frac{b}{(s-a)^2 - b^2}$	$s > a + \|b\|$

#	Function / Property Name	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	Condition
18	Damped Hyperbolic Cosine	$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$	$s > a + \lvert b \rvert$
19	Frequency Integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$	$\lim_{t\to 0^+} \frac{f(t)}{t}$ exists
20	First Derivative	$f'(t)$	$sF(s) - f(0)$	—
21	Second Derivative	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	—
22	Time Integration	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	$s > 0$
23	Time Convolution	$(f * g)(t)$	$F(s)G(s)$	—
24	General Power Function	$t^p \ (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
25	Fractional Square Root	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	$s > 0$

Tip

Using the table:

- **Forward transform:** find $f(t)$ in the left column, read $F(s)$ in the middle.
- **Inverse transform:** find $F(s)$ in the middle column, read $f(t)$ on the left.

5 Theorems and Important Properties

5.1 Theorem 1 — Linearity

Theorem 4.1 — Linearity Property

$$\mathcal{L}\{a\,f(t) + b\,g(t)\} = a\,F(s) + b\,G(s)$$

Split a sum and transform each piece separately; constants factor out.

Example 1: Find $\mathcal{L}\{5t^3 - 2e^{4t} + 7\}$.

$$= 5\,\mathcal{L}\{t^3\} - 2\,\mathcal{L}\{e^{4t}\} + 7\,\mathcal{L}\{1\} = 5 \cdot \frac{6}{s^4} - 2 \cdot \frac{1}{s-4} + 7 \cdot \frac{1}{s}$$

$$\boxed{= \frac{30}{s^4} - \frac{2}{s-4} + \frac{7}{s}}$$

Example 2: Find $\mathcal{L}\{3\sin 2t + 4\cos 2t\}$.

$$= 3 \cdot \frac{2}{s^2+4} + 4 \cdot \frac{s}{s^2+4}$$

$$\boxed{= \frac{4s+6}{s^2+4}}$$

Example 3: Find $\mathcal{L}\{(2t-3)^2\}$.

Expand first: $(2t-3)^2 = 4t^2 - 12t + 9$.

$$= 4 \cdot \frac{2}{s^3} - 12 \cdot \frac{1}{s^2} + 9 \cdot \frac{1}{s}$$

$$\boxed{= \frac{8}{s^3} - \frac{12}{s^2} + \frac{9}{s}}$$

5.2 Theorem 2 — The First Shifting Theorem

Theorem 4.2 — First Shifting Theorem

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Multiplying $f(t)$ by e^{at} replaces every s in $F(s)$ with $(s-a)$.

Three-step method:

1. Find $F(s) = \mathcal{L}\{f(t)\}$ ignoring the e^{at} factor
2. Identify a
3. Replace every s in $F(s)$ with $(s-a)$

Example 1: Find $\mathcal{L}\{e^{3t} \sin 2t\}$.

$f(t) = \sin 2t \Rightarrow F(s) = \dfrac{2}{s^2+4}$. With $a = 3$, replace $s \to s-3$:

$$\boxed{\mathcal{L}\{e^{3t} \sin 2t\} = \frac{2}{(s-3)^2+4}}$$

Example 2: Find $\mathcal{L}\{e^{-2t} \cos 5t\}$.

$F(s) = \dfrac{s}{s^2+25}$, $a = -2$, replace $s \to s+2$:

$$\boxed{\mathcal{L}\{e^{-2t} \cos 5t\} = \frac{s+2}{(s+2)^2+25}}$$

Example 3: Find $\mathcal{L}\{t^2 e^{-4t}\}$.

$F(s) = \dfrac{2}{s^3}$, $a = -4$, replace $s \to s+4$:

$$\boxed{\mathcal{L}\{t^2 e^{-4t}\} = \frac{2}{(s+4)^3}}$$

5.3 Theorem 3 — Transform of Derivatives

Theorem 4.3 — Laplace Transform of Derivatives

$$\mathcal{L}\{y'(t)\} = s\,Y(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - s\,y(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - s\,y'(0) - y''(0)$$

Memory Aid

Start with $s^nY(s)$, then subtract correction terms using initial values, reducing the power of s by 1 each time:

$$\mathcal{L}\{y^{(n)}\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$$

Derivation sketch for y': Integration by parts on $\int_0^\infty e^{-st}y'\,dt$ with $u = e^{-st}$, $dv = y'\,dt$ gives $[e^{-st}y]_0^\infty + s\int_0^\infty e^{-st}y\,dt = 0 - y(0) + sY(s)$.

Example 1: Write $\mathcal{L}\{y''\}$ for $y(0) = 3$, $y'(0) = -1$:

$$\mathcal{L}\{y''\} = s^2Y(s) - 3s + 1$$

Example 2: Write $\mathcal{L}\{y''\}$ for $y(0) = 0$, $y'(0) = 0$:

$$\mathcal{L}\{y''\} = s^2Y(s)$$

Example 3: Write $\mathcal{L}\{y''\}$ for $y(0) = -2$, $y'(0) = 5$:

$$\mathcal{L}\{y''\} = s^2Y(s) + 2s - 5$$

6 Inverse Laplace Transforms

6.1 What Is the Inverse?

Definition 4.2 — Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Also linear:

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\,f(t) + b\,g(t)$$

Use the transform table **in reverse** — recognise the form of $F(s)$ and read off $f(t)$.

Example 1: $\mathcal{L}^{-1}\left\{\dfrac{1}{s+5}\right\}$

Row 4 with $a = -5$: $\boxed{e^{-5t}}$

Example 2: $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\}$

Row 5 with $b = 3$: $\boxed{\sin 3t}$

Example 3: $\mathcal{L}^{-1}\left\{\dfrac{s-2}{(s-2)^2+16}\right\}$

Row 11 with $a = 2$, $b = 4$: $\boxed{e^{2t}\cos 4t}$

6.2 Partial Fractions — Breaking Down Complicated Expressions

What Is Partial Fraction Decomposition?

Writing a complicated rational function as a sum of simpler ones, e.g.:

$$\frac{5s+3}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

Requirement: The degree of the numerator must be strictly *less than* the degree of the denominator. Perform polynomial long division first if it is not.

6.2.1 The 6-Step Solution Framework

To solve any partial fraction problem, follow these systematic steps:

1. **Check the Degree:** Ensure the numerator's highest power is smaller than the denominator's.
2. **Factor the Denominator:** Break it down completely into linear or irreducible quadratic factors.
3. **Set Up the Template:** Write out the generic fractions with unknown constants (A, B, C) based on the factor case.
4. **Clear the Fractions:** Multiply the entire equation by the original denominator to get a flat, single-line equation.
5. **Solve for Constants:** Use strategic substitution (setting s to roots) or compare coefficients.
6. **Write the Final Answer:** Substitute the constants back into your template.

6.2.2 Case 1 — Distinct Linear Factors

Form:

$$\frac{P(s)}{(s-a)(s-b)\cdots} = \frac{A}{s-a} + \frac{B}{s-b} + \cdots$$

Cover-up rule for A: set $s = a$ in everything except the factor $(s-a)$.

Example 1

Find $\mathcal{L}^{-1}\left\{\dfrac{3}{(s-1)(s+2)}\right\}$.

Step 1: Check the Degree

- The highest power of s in the numerator is 0 (since it is a constant, 3).
- The highest power of s in the denominator is 2 (from $s \cdot s = s^2$).
- **Condition Met:** The numerator degree (0) is strictly less than the denominator degree (2).

Step 2: Factor the Denominator

- The denominator $(s-1)(s+2)$ is already fully factored into two distinct linear terms.

Step 3: Set Up the Template

Assign a unique constant variable (A, B) to sit over each unique linear factor:

$$\frac{3}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

Step 4: Clear Fractions / Use Cover-Up Rule

Because these are distinct linear factors, we can use the **Cover-Up Method** to rapidly isolate each constant.

To isolate A:

1. Identify the denominator under A, which is $(s-1)$.
2. Find its root by setting it to zero: $s - 1 = 0 \implies \mathbf{s = 1}$.
3. Go back to the original function, completely "cover up" or remove the $(s-1)$ factor, and plug $s = 1$ into what remains:

$$A = \left. \frac{3}{\blacksquare \quad (s+2)} \right|_{s=1} = \frac{3}{1+2} = \frac{3}{3} = \mathbf{1}$$

To isolate B:

1. Identify the denominator under B, which is $(s+2)$.
2. Find its root by setting it to zero: $s + 2 = 0 \implies \mathbf{s = -2}$.
3. Go back to the original function, completely "cover up" the $(s+2)$ factor, and plug $s = -2$ into what remains:

$$B = \left. \frac{3}{(s-1) \quad \blacksquare} \right|_{s=-2} = \frac{3}{-2-1} = \frac{3}{-3} = -\mathbf{1}$$

Step 5: Write the Decomposed Fraction

Substitute your solved values ($A = 1$ and $B = -1$) back into your original template step:

$$\frac{3}{(s-1)(s+2)} = \frac{1}{s-1} - \frac{1}{s+2}$$

Step 6: Apply the Inverse Laplace Transform

Use the linear standard Laplace pairs lookup table: $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$. * For $\frac{1}{s-1}$, our $a = 1 \implies \mathbf{e^t}$
* For $\frac{1}{s+2}$, our $a = -2 \implies \mathbf{e^{-2t}}$

$$\boxed{f(t) = e^t - e^{-2t}}$$

Example 2

Find $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{(s+1)(s-2)(s+3)}\right\}$.

Step 1: Check the Degree

- Numerator degree: 1 (from $2s^1$).
- Denominator degree: 3 (from $s \cdot s \cdot s = s^3$).
- **Condition Met:** $1 < 3$.

Step 2: Factor the Denominator

- The denominator $(s+1)(s-2)(s+3)$ is already fully factored.

Step 3: Set Up the Template

Create three fractions for the three distinct linear factors:

$$\frac{2s+3}{(s+1)(s-2)(s+3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+3}$$

Step 4: Solve for Constants (Cover-Up Method)

To isolate A (Set denominator $s + 1 = 0 \implies \mathbf{s = -1}$):

Cover up $(s+1)$ in the original expression and evaluate at $s = -1$:

$$A = \frac{2(-1)+3}{([-1]-2)([-1]+3)} = \frac{-2+3}{(-3)(2)} = -\frac{\mathbf{1}}{\mathbf{6}}$$

To isolate B (Set denominator $s - 2 = 0 \implies \mathbf{s = 2}$):

Cover up $(s-2)$ in the original expression and evaluate at $s=2$:

$$B = \frac{2(2)+3}{(+1)(+3)} = \frac{4+3}{(3)(5)} = \frac{\mathbf{7}}{\mathbf{15}}$$

To isolate C (Set denominator $s+3=0 \implies \mathbf{s=-3}$):

Cover up $(s+3)$ in the original expression and evaluate at $s=-3$:

$$C = \frac{2(-3)+3}{([-3]+1)(-2)} = \frac{-6+3}{(-2)(-5)} = \frac{-3}{10} = -\frac{\mathbf{3}}{\mathbf{10}}$$

Step 5: Write the Decomposed Fraction

$$\frac{2s+3}{(s+1)(s-2)(s+3)} = -\frac{1}{6}\left(\frac{1}{s+1}\right) + \frac{7}{15}\left(\frac{1}{s-2}\right) - \frac{3}{10}\left(\frac{1}{s+3}\right)$$

Step 6: Apply the Inverse Laplace Transform

Apply the lookup transform rule $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ directly to each independent part:

$$\boxed{f(t) = -\frac{1}{6}e^{-t} + \frac{7}{15}e^{2t} - \frac{3}{10}e^{-3t}}$$

6.2.3 Case 2 — Repeated Factors

A repeated factor $(s-a)^n$ requires n terms in the template setup:

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

Example 1

Find $\mathcal{L}^{-1}\left\{\dfrac{4}{s^2(s+2)}\right\}$.

Step 1 & 2: Check Degree and Factor

- Numerator degree is 0; Denominator degree is 3 ($s^2 \cdot s = s^3$). Condition met.
- The term s^2 represents a **repeated linear factor** (s occurs twice).

Step 3: Set Up the Template

For repeated factors, you must build up the powers sequentially until you hit the highest exponent:

$$\frac{4}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

Step 4: Clear the Fractions

Multiply every single term across the entire equation by the LCD (lowest common denominator), which is $\mathbf{s^2(s+2)}$:

$$4 = A \cdot s(s+2) + B \cdot (s+2) + C \cdot s^2$$

Step 5: Solve for Constants Using Strategic Substitution

We can pick any values for s to eliminate terms and find our unknowns. Let's use the natural roots first.

Choose $s = 0$ (Eliminates A and C terms):

$$4 = A(0)(0+2) + B(0+2) + C(0)^2$$

$$4 = 2B \implies \mathbf{B = 2}$$

Choose $s = -2$ (Eliminates A and B terms):

$$4 = A(-2)(-2+2) + B(-2+2) + C(-2)^2$$

$$4 = C(4) \implies 4 = 4C \implies \mathbf{C = 1}$$

Find A by Equating Coefficients:

Let us look at the s^2 terms on both sides of our cleared equation. * Left side: There are 0 occurrences of s^2. * Right side: If you distribute out the terms, you get $As^2 + 2As + Bs + 2B + Cs^2$. Collecting s^2 gives $(A+C)s^2$.

Set them equal:

$$0 = A + C$$

Since we know $C = 1$:

$$0 = A + 1 \implies \mathbf{A = -1}$$

Step 6: Rebuild and Transform

Substitute constants back into the setup:

$$\frac{4}{s^2(s+2)} = -\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s+2}$$

Apply Laplace inverse pairs ($\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$, $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$, and $\mathcal{L}^{-1}\{\frac{1}{s+2}\} = e^{-2t}$):

$$\boxed{f(t) = -1 + 2t + e^{-2t}}$$

Example 2

Find $\mathcal{L}^{-1}\left\{\dfrac{s+3}{(s+1)^2(s-2)}\right\}$.

Step 1 to 3: Setup Template

The denominator contains a repeated linear factor $(s+1)^2$ and a distinct linear factor $(s-2)$.

$$\frac{s+3}{(s+1)^2(s-2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-2}$$

Step 4: Clear Fractions

Multiply the entire equation by the LCD $(\mathbf{s+1})^{\mathbf{2}}(\mathbf{s-2})$:

$$s + 3 = A(s+1)(s-2) + B(s-2) + C(s+1)^2$$

Step 5: Solve for Constants

Choose $s = -1$:

$$-1+3 = A(0)(-3) + B(-1-2) + C(0)^2$$

$$2 = -3B \implies \mathbf{B} = -\frac{\mathbf{2}}{\mathbf{3}}$$

Choose $s = 2$:

$$2+3 = A(3)(0) + B(0) + C(2+1)^2$$

$$5 = 9C \implies \mathbf{C} = \frac{\mathbf{5}}{\mathbf{9}}$$

Equate the highest power coefficients (s^2 terms):

- Left Side: 0
- Right Side: Expanding gives $A(s^2 - s - 2) + B(s-2) + C(s^2 + 2s + 1)$. Look only at $s^2 \implies As^2 + Cs^2$.

$$0 = A + C \implies A = -C \implies \mathbf{A} = -\frac{\mathbf{5}}{\mathbf{9}}$$

Step 6: Transform

$$\mathcal{L}^{-1}\left\{-\frac{5}{9}\left(\frac{1}{s+1}\right) - \frac{2}{3}\left(\frac{1}{(s+1)^2}\right) + \frac{5}{9}\left(\frac{1}{s-2}\right)\right\}$$

Note: $\mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$ uses the Frequency Shift Theorem $\mathcal{L}^{-1}\{G(s-a)\} = e^{at}g(t)$. Since $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$, shifting by $a = -1$ yields te^{-t}.

$$\boxed{f(t) = -\frac{5}{9}e^{-t} - \frac{2}{3}te^{-t} + \frac{5}{9}e^{2t}}$$

6.2.4 Case 3 — Irreducible Quadratic Factors

Completing the Square Formula

$$s^2 + ps + q = \left(s + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)$$

Match this to the standard form $(s-a)^2 + b^2$ where $a = -p/2$ and $b^2 = q - p^2/4$.

Example 1

Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}$.

Step 1: Analyze the Denominator

Can we factor $s^2 + 4s + 5$ into real numbers? Let's check the discriminant ($b^2 - 4ac$):

$$4^2 - 4(1)(5) = 16 - 20 = -4$$

Because the discriminant is negative, this is an **irreducible quadratic**. We must complete the square.

Step 2: Complete the Square

1. Take the middle coefficient (4), divide by 2 $\implies$ 2, then square it $\implies$ **4**.
2. Add and subtract this value inside the expression:

$$s^2 + 4s + 5 = (s^2 + 4s + 4) - 4 + 5$$

3.

3. Collapse the perfect square trinomial:

$$\mathbf{(s+2)^2 + 1}$$

Step 3: Transform Using Shift Properties

Rewrite the inverse Laplace transform problem:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

This fits the First Shifting Theorem model:

$$\mathcal{L}^{-1}\{G(s-a)\} = e^{at}g(t)$$

- **Identify the base function** without the shift: $\frac{1}{s^2+1^2}$, which equals $\sin(1t)$.
- **Identify the shift parameter**: $(s+2) \implies s-(-2)$, so $a=-2$.
- **Attach the exponential shift multiplier** e^{-2t} directly to the sine function:

$$\boxed{f(t) = e^{-2t}\sin t}$$

Example 2

Find $\mathcal{L}^{-1}\left\{\frac{2s+8}{s^2+4s+8}\right\}$.

Step 1: Complete the Square in Denominator

$$s^2+4s+8 = (s^2+4s+4)-4+8 = \mathbf{(s+2)^2+4}$$

Our frequency shift target is $(s+2)$, and our frequency parameter is:

$$\omega^2 = 4 \implies \ = \mathbf{2}$$

Step 2: Adjust Numerator to Match Shift

Every instance of s in the numerator must be transformed into an explicit $(s+2)$ grouping to satisfy shifting laws:

$$2s+8 = 2(s+2) + \text{remainder}$$

Distribute to verify: $2s + 4$. To get back to $+8$, we need to add another 4:

$$2s + 8 = \mathbf{2(s + 2) + 4}$$

Step 3: Break up into Two Independent Fractions

$$\frac{2(s+2)+4}{(s+2)^2+4} = \frac{2(s+2)}{(s+2)^2+4} + \frac{4}{(s+2)^2+4}$$

Pull out constant multipliers to isolate base formulas:

$$= 2 \cdot \left[\frac{s+2}{(s+2)^2+2^2}\right] + 2 \cdot \left[\frac{2}{(s+2)^2+2^2}\right]$$

Step 4: Apply Inverse Transformations

- **Form 1:** $\frac{s}{s^2+\omega^2} \to \cos(\omega t)$. Shifted by $a = -2 \implies \mathbf{e^{-2t}}\cos(\mathbf{2t})$.
- **Form 2:** $\frac{\omega}{s^2+\omega^2} \to \sin(\omega t)$. Shifted by $a = -2 \implies \mathbf{e^{-2t}}\sin(\mathbf{2t})$.

$$\boxed{f(t) = 2e^{-2t}\cos 2t + 2e^{-2t}\sin 2t = 2e^{-2t}(\cos 2t + \sin 2t)}$$

Example 3

Find $\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s-1)(s^2+2s+5)}\right\}$.

Step 1: Complete the Square

The quadratic factor $s^2 + 2s + 5$ has a negative discriminant. Complete its square:

$$s^2 + 2s + 5 = (s^2 + 2s + 1) - 1 + 5 = \mathbf{(s+1)^2 + 4}$$

Step 2: Set up Template

An irreducible quadratic factor requires a linear numerator template $(Bs + C)$:

$$\frac{s+1}{(s-1)[(s+1)^2+4]} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

Step 3: Clear Fractions

Multiply by the lowest common denominator (LCD) $(s-1)(s^2+2s+5)$:

$$s + 1 = A(s^2 + 2s + 5) + (Bs + C)(s - 1)$$

Step 4: Solve for Constants

Solve for A using Cover-Up (Set $s = 1$):

$$1 + 1 = A(1^2 + 2(1) + 5) + (B(1) + C)(0)$$

$$2 = 8A \implies A = \frac{2}{8} \implies \mathbf{A} = \frac{\mathbf{1}}{\mathbf{4}}$$

Solve for B by Equating s^2 Coefficients:

- **Left side:** 0
- **Right side:** $As^2 + Bs^2 \implies A + B$

$$0 = A + B \implies B = -A \implies \mathbf{B} = -\frac{\mathbf{1}}{\mathbf{4}}$$

Solve for C by Equating Constant Values (Set $s = 0$):

$$0 + 1 = A(5) + (B(0) + C)(0 - 1)$$

$$1 = 5A - C$$

Substitute $A = \frac{1}{4}$:

$$1 = \frac{5}{4} - C \implies C = \frac{5}{4} - 1 \implies \mathbf{C} = \frac{\mathbf{1}}{\mathbf{4}}$$

(Note: Expanding and equating coefficients yields $B = -\dfrac{1}{4}$ and $C = \dfrac{3}{4}$ in the original unshifted formulation).

Step 5: Structure the Quadratic Term for Shifting

Our quadratic piece is:

$$\frac{-\frac{1}{4}s + \frac{3}{4}}{(s+1)^2 + 4}$$

Because our denominator is shifted to $(s+1)$, we must shift our numerator s to $(s+1)$ as well:

$$-\frac{1}{4}s + \frac{3}{4} = -\frac{1}{4}(s+1) + \text{remainder}$$

Distributing gives $-\frac{1}{4}s - \frac{1}{4}$. To achieve our target value of $+\frac{3}{4}$, we must add $\frac{4}{4}$ (or 1):

$$= -\frac{1}{4}(s+1) + 1$$

Now substitute this adjusted layout back over the denominator:

$$= -\frac{1}{4} \cdot \frac{s+1}{(s+1)^2 + 4} + \frac{1}{4} \cdot \frac{2}{(s+1)^2 + 4}$$

Step 6: Take the Final Inverse Laplace Transform

Combine all three components (A component + rewritten quadratic components):

$$\boxed{f(t) = \frac{1}{4}e^{t} - \frac{1}{4}e^{-t}\cos 2t + \frac{1}{4}e^{-t}\sin 2t}$$

7 Solving Differential Equations Using Laplace Transforms

7.1 The Four-Step Procedure

Four-Step Procedure for an IVP

Given $y'' + ay' + by = g(t)$, $y(0) = y_0$, $y'(0) = y_1$:

STEP 1 — Transform every term in the ODE.

STEP 2 — Substitute the initial conditions $y(0)$ and $y'(0)$.

STEP 3 — Solve for $Y(s)$ algebraically.

STEP 4 — Invert using partial fractions and the table to find $y(t)$.

7.2 Solving First-Order IVPs

7.2.1 Example 1

Problem: Solve $y' - 3y = 0$ given the initial boundary condition $y(0) = 2$.

- **Step 1: Transform every term**

$$\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$[sY(s) - y(0)] - 3Y(s) = 0$$

- **Step 2: Substitute initial conditions**

$$sY(s) - 2 - 3Y(s) = 0$$

- **Step 3: Solve for $Y(s)$ algebraically** Group the terms containing $Y(s)$ together and move constants to the right side:

$$(s-3)Y(s) - 2 = 0 \implies (s-3)Y(s) = 2$$

$$Y(s) = \frac{2}{s-3}$$

- **Step 4: Invert to find** $y(t)$ Factor out the constant numerator and use the standard pair $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ where $a = 3$:

$$y(t) = 2 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$\boxed{y(t) = 2e^{3t}}$$

7.2.2 Example 2

Problem: Solve $y' + 4y = 8$ given the initial condition $y(0) = 1$.

- **Step 1 & 2: Transform and Substitute**

$$\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{8\}$$

$$[sY(s) - y(0)] + 4Y(s) = \frac{8}{s}$$

$$sY(s) - 1 + 4Y(s) = \frac{8}{s}$$

- **Step 3: Solve for** $Y(s)$ **algebraically** Isolate the $Y(s)$ terms on the left by adding 1 to both sides:

$$(s+4)Y(s) - 1 = \frac{8}{s} \implies (s+4)Y(s) = \frac{8}{s} + 1$$

Find a common denominator for the right-hand side expressions:

$$(s+4)Y(s) = \frac{8}{s} + \frac{s}{s} \implies (s+4)Y(s) = \frac{s+8}{s}$$

Divide both sides by $(s+4)$:

$$Y(s) = \frac{s+8}{s(s+4)}$$

- **Step 4: Partial Fractions and Inversion** Set up the linear decomposition template:

$$\frac{s+8}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

  - Use the cover-up method to find A (set $s = 0$):

$$A = \left.\frac{s+8}{s+4}\right|_{s=0} = \frac{0+8}{0+4} = \mathbf{2}$$

  - Use the cover-up method to find B (set $s = -4$):

$$B = \left.\frac{s+8}{s}\right|_{s=-4} = \frac{-4+8}{-4} = \frac{4}{-4} = -\mathbf{1}$$

  Substitute the constants back into the transform function and apply the inverse pairs:

$$Y(s) = \frac{2}{s} - \frac{1}{s+4}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\boxed{y(t) = 2 - e^{-4t}}$$

7.2.3 Example 3

Problem: Solve $y' + 2y = e^{-t}$ given the initial condition $y(0) = 0$.

- **Step 1 & 2: Transform and Substitute**

$$[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}$$

$$sY(s) - 0 + 2Y(s) = \frac{1}{s+1}$$

- **Step 3: Solve for** $Y(s)$ algebraically

$$(s+2)Y(s) = \frac{1}{s+1} \implies Y(s) = \frac{1}{(s+1)(s+2)}$$

- **Step 4: Partial Fractions and Inversion**

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

  – Find A (cover up $s+1$ and set $s=-1$):

$$A = \left.\frac{1}{s+2}\right|_{s=-1} = \frac{1}{-1+2} = \mathbf{1}$$

  – Find B (cover up $s+2$ and set $s=-2$):

$$B = \left.\frac{1}{s+1}\right|_{s=-2} = \frac{1}{-2+1} = \frac{1}{-1} = -\mathbf{1}$$

Rebuild the clean inverse equations:

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\boxed{y(t) = e^{-t} - e^{-2t}}$$

7.3 Solving Second-Order IVPs

7.3.1 Example 1

Problem: Solve $y'' + 5y' + 6y = 0$ given $y(0) = 2$ and $y'(0) = -1$.

- **Step 1: Transform every term**

$$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\left[s^2Y(s) - s \cdot y(0) - y'(0)\right] + 5\left[sY(s) - y(0)\right] + 6Y(s) = 0$$

- **Step 2: Substitute initial conditions**

$$\left[s^2Y(s) - s(2) - (-1)\right] + 5\left[sY(s) - 2\right] + 6Y(s) = 0$$

$$s^2Y(s) - 2s + 1 + 5sY(s) - 10 + 6Y(s) = 0$$

- **Step 3: Solve for** $Y(s)$ algebraically Group all terms with a $Y(s)$ multiplier and group the remaining constant terms:

$$\left(s^2 + 5s + 6\right)Y(s) - 2s + 1 - 10 = 0$$

$$\left(s^2 + 5s + 6\right)Y(s) - 2s - 9 = 0$$

Move non-$Y(s)$ terms to the right side and factor the denominator polynomial:

$$\left(s^2 + 5s + 6\right)Y(s) = 2s + 9$$

$$Y(s) = \frac{2s+9}{s^2+5s+6} = \frac{2s+9}{(s+2)(s+3)}$$

- **Step 4: Partial Fractions and Inversion**

$$\frac{2s+9}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

  - Find A (set $s = -2$):

$$A = \left.\frac{2s+9}{s+3}\right|_{s=-2} = \frac{2(-2)+9}{-2+3} = \frac{-4+9}{1} = \mathbf{5}$$

  - Find B (set $s = -3$):

$$B = \left.\frac{2s+9}{s+2}\right|_{s=-3} = \frac{2(-3)+9}{-3+2} = \frac{-6+9}{-1} = \frac{3}{-1} = \mathbf{-3}$$

Invert the resulting distinct linear expressions:

$$Y(s) = \frac{5}{s+2} - \frac{3}{s+3} \implies \boxed{y(t) = 5e^{-2t} - 3e^{-3t}}$$

7.3.2 Example 2

Problem: Solve $y'' + y = \sin t$ given $y(0) = 0$ and $y'(0) = 0$.

- **Step 1 & 2: Transform and Substitute**

$$\left[s^2 Y(s) - s \cdot y(0) - y'(0)\right] + Y(s) = \frac{1}{s^2+1}$$

$$\left[s^2 Y(s) - s(0) - 0\right] + Y(s) = \frac{1}{s^2+1} \implies s^2 Y(s) + Y(s) = \frac{1}{s^2+1}$$

- **Step 3: Solve for** $Y(s)$ algebraically

$$\left(s^2+1\right) Y(s) = \frac{1}{s^2+1} \implies Y(s) = \frac{1}{\left(s^2+1\right)\left(s^2+1\right)} = \frac{1}{(s^2+1)^2}$$

- **Step 4: Invert using specialized table transforms** When a linear partial fraction template cannot break down repeated irreducible terms easily, we use specialized tables:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} = \frac{\sin(\omega t) - \omega t\cos(\omega t)}{2\omega^3}$$

Setting our frequency parameter $\omega = 1$:

$$\boxed{y(t) = \frac{\sin t - t\cos t}{2}}$$

> **Resonance**
>
> The presence of the unbound linearly growing factor t in front of the cosine term represents **resonance**. This always occurs physically when the external forcing frequency matches the natural frequency of the differential equation system.

7.3.3 Example 3

Problem: Solve $y'' - 4y' + 4y = e^{2t}$ given $y(0) = 0$ and $y'(0) = 0$.

- **Step 1 & 2: Transform and Substitute**

$$\left[s^2Y(s) - s(0) - 0\right] - 4\left[sY(s) - 0\right] + 4Y(s) = \frac{1}{s-2}$$

$$s^2Y(s) - 4sY(s) + 4Y(s) = \frac{1}{s-2}$$

- **Step 3: Solve for $Y(s)$** algebraically Factor out the characteristic polynomial grouping from the transform expression:

$$\left(s^2 - 4s + 4\right)Y(s) = \frac{1}{s-2}$$

Recognize that the polynomial is a perfect square trinomial: $(s^2 - 4s + 4) = (s-2)^2$.

$$(s-2)^2Y(s) = \frac{1}{s-2} \implies Y(s) = \frac{1}{(s-2)^1 \cdot (s-2)^2} = \frac{1}{(s-2)^3}$$

- **Step 4: Invert using the Repeated Shift Rule** Recall the repeated frequency shift standard pair lookup form:

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = t^n e^{at}$$

To match our target function exponent of 3, we need $n + 1 = 3 \implies n = 2$. This requires a numerator value of $2! = 2$. Multiply by $\frac{1}{2}$ externally to balance the change:

$$Y(s) = \frac{1}{2} \cdot \left[\frac{2!}{(s-2)^3} \right]$$

$$\boxed{y(t) = \frac{1}{2} t^2 e^{2t}}$$

7.3.4 Example 4 (Full working)

Problem: Solve $y'' + 3y' + 2y = 4t$ given $y(0) = 1$ and $y'(0) = 0$.

- **Step 1: Transform every term**

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{4t\}$$

$$\left[s^2 Y(s) - s \cdot y(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{4}{s^2}$$

- **Step 2: Substitute initial conditions**

$$\left[s^2 Y(s) - s(1) - 0\right] + 3\left[sY(s) - 1\right] + 2Y(s) = \frac{4}{s^2}$$

$$s^2 Y(s) - s + 3sY(s) - 3 + 2Y(s) = \frac{4}{s^2}$$

- **Step 3: Solve for $Y(s)$** algebraically Group the $Y(s)$ expressions together and move remaining polynomials to the right:

$$\left(s^2 + 3s + 2\right) Y(s) - s - 3 = \frac{4}{s^2}$$

$$\left(s^2 + 3s + 2\right) Y(s) = \frac{4}{s^2} + s + 3$$

Combine the right-hand terms under a common denominator of s^2:

$$\frac{4}{s^2} + \frac{s \cdot s^2}{s^2} + \frac{3 \cdot s^2}{s^2} = \frac{4 + s^3 + 3s^2}{s^2} = \frac{s^3 + 3s^2 + 4}{s^2}$$

Divide by the factored polynomial coefficient $(s^2 + 3s + 2) = (s + 1)(s + 2)$:

$$Y(s) = \frac{s^3 + 3s^2 + 4}{s^2(s + 1)(s + 2)}$$

- **Step 4: Partial Fraction Decomposition** Set up the template covering the distinct linear terms along with the repeated linear factor:

$$\frac{s^3 + 3s^2 + 4}{s^2(s + 1)(s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 1} + \frac{D}{s + 2}$$

  - **Find** B (Multiply by s^2 and evaluate at $s = 0$):

$$B = \left.\frac{s^3 + 3s^2 + 4}{(s + 1)(s + 2)}\right|_{s=0} = \frac{0 + 0 + 4}{(1)(2)} = \frac{4}{2} = \mathbf{2}$$

  - **Find** C (Multiply by $s + 1$ and evaluate at $s = -1$):

$$C = \left.\frac{s^3 + 3s^2 + 4}{s^2(s + 2)}\right|_{s=-1} = \frac{(-1)^3 + 3(-1)^2 + 4}{(-1)^2(-1 + 2)} = \frac{-1 + 3 + 4}{(1)(1)} = \frac{6}{1} = \mathbf{6}$$

  - **Find** D (Multiply by $s + 2$ and evaluate at $s = -2$):

$$D = \left.\frac{s^3 + 3s^2 + 4}{s^2(s + 1)}\right|_{s=-2} = \frac{(-2)^3 + 3(-2)^2 + 4}{(-2)^2(-2 + 1)} = \frac{-8 + 12 + 4}{(4)(-1)} = \frac{8}{-4} = -\mathbf{2}$$

  - **Find** A (Equate the coefficients of the highest power, s^3): Clear fractions to get the flat single-line numerator expression:

Clear fractions to get the flat, single-line numerator expression:

$$s^3 + 3s^2 + 4 = As(s + 1)(s + 2) + B(s + 1)(s + 2) + Cs^2(s + 2) + Ds^2(s + 1)$$

Look only at the expanded parts that generate an s^3 term:

$$s^3 = As^3 + Cs^3 + Ds^3 \implies 1 = A + C + D$$

Substitute our known values for $C = 6$ and $D = -2$ into this matching relation:

$$1 = A + 6 + (-2) \implies 1 = A + 4 \implies A = 1 - 4 = -\mathbf{3}$$

Reassemble the broken terms and take the term-by-term inverse transform:

$$Y(s) = -\frac{3}{s} + \frac{2}{s^2} + \frac{6}{s+1} - \frac{2}{s+2}$$

$$\boxed{y(t) = -3 + 2t + 6e^{-t} - 2e^{-2t}}$$

8 Solving Simultaneous Differential Equations

8.1 The Approach

When dealing with a system of coupled differential equations, we transform **both** equations simultaneously into the frequency domain. This completely eliminates the derivatives, converting the system of calculus differential equations into a standard system of two linear algebraic equations. We can then isolate our target variables $X(s)$ and $Y(s)$ using basic substitution or elimination methods before performing the inverse transforms.

8.2 Example 1

Problem: Solve the system of coupled differential equations given the initial conditions $x(0) = 1$ and $y(0) = 0$:

$$x' = x + y$$

$$y' = -2x + 4y$$

8.2.1 Step 1: Transform both equations into the frequency domain

Apply the Laplace operator to every term:

$$\mathcal{L}\{x'\} = \mathcal{L}\{x\} + \mathcal{L}\{y\} \implies [sX(s) - x(0)] = X(s) + Y(s)$$

$$\mathcal{L}\{y'\} = -2\mathcal{L}\{x\} + 4\mathcal{L}\{y\} \implies [sY(s) - y(0)] = -2X(s) + 4Y(s)$$

Substitute the initial conditions $x(0) = 1$ and $y(0) = 0$:

$$sX(s) - 1 = X(s) + Y(s)$$

$$sY(s) - 0 = -2X(s) + 4Y(s)$$

Move all variables $X(s)$ and $Y(s)$ to the left side and isolate the constants on the right:

$$sX(s) - X(s) - Y(s) = 1 \implies (s-1)X(s) - Y(s) = 1 \tag{i}$$

$$2X(s) + sY(s) - 4Y(s) = 0 \implies 2X(s) + (s-4)Y(s) = 0 \tag{ii}$$

8.2.2 Step 2: Isolate $X(s)$ using equation (ii)

Using equation (ii), move the $Y(s)$ block to the right side:

$$2X(s) = -(s-4)Y(s)$$

Divide both sides by 2:

$$X(s) = -\frac{(s-4)Y(s)}{2} \tag{iii}$$

8.2.3 Step 3: Substitute equation (iii) into equation (i)

Replace $X(s)$ in equation (i) with our new fractional identity:

$$(s-1)\left[-\frac{(s-4)Y(s)}{2}\right] - Y(s) = 1$$

To clear the fraction, multiply every single term across the entire equation by 2:

$$-(s-1)(s-4)Y(s) - 2Y(s) = 2$$

Factor out $-Y(s)$ from the left-hand side to keep the algebra organized:

$$-Y(s) \cdot [(s-1)(s-4) + 2] = 2$$

Expand the polynomial inner brackets:

$$(s-1)(s-4) = s^2 - 4s - s + 4 = s^2 - 5s + 4$$

Substitute this back inside the bracket tracking group:

$$-Y(s) \cdot \left[s^2 - 5s + 4 + 2\right] = 2 \implies -Y(s) \cdot \left(s^2 - 5s + 6\right) = 2$$

Multiply the whole equation by -1 and factor the quadratic expression $(s^2-5s+6 = (s-2)(s-3))$:

$$Y(s) \cdot (s-2)(s-3) = -2$$

$$Y(s) = \frac{-2}{(s-2)(s-3)}$$

8.2.4 Step 4: Perform Partial Fraction Decomposition for $Y(s)$

$$\frac{-2}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

- **Find** A (cover up $s-2$ and set $s=2$):

$$A = \left.\frac{-2}{s-3}\right|_{s=2} = \frac{-2}{2-3} = \frac{-2}{-1} = \mathbf{2}$$

- **Find** B (cover up $s-3$ and set $s=3$):

$$B = \left.\frac{-2}{s-2}\right|_{s=3} = \frac{-2}{3-2} = \frac{-2}{1} = -\mathbf{2}$$

Reassemble and apply the standard linear inverse transform lookup pair:

$$Y(s) = \frac{2}{s-2} - \frac{2}{s-3}$$

$$\boxed{y(t) = 2e^{2t} - 2e^{3t}}$$

8.2.5 Step 5: Solve for $x(t)$

Substitute our values for $A=2$ and $B=-2$ back into equation (iii) to find the explicit equation for $X(s)$:

$$X(s) = -\frac{(s-4)}{2} \cdot \left[\frac{-2}{(s-2)(s-3)}\right]$$

Cancel the 2 in the denominator with the -2 in the numerator, which flips the signs of our front bracket from $(s-4)$ to $-(s-4) = (4-s)$:

$$X(s) = \frac{4-s}{(s-2)(s-3)}$$

Set up partial fractions for $X(s)$:

$$\frac{4-s}{(s-2)(s-3)} = \frac{C}{s-2} + \frac{D}{s-3}$$

- **Find** C (set $s = 2$):

$$C = \left.\frac{4-s}{s-3}\right|_{s=2} = \frac{4-2}{2-3} = \frac{2}{-1} = -\mathbf{2}$$

- **Find** D (set $s = 3$):

$$D = \left.\frac{4-s}{s-2}\right|_{s=3} = \frac{4-3}{3-2} = \frac{1}{1} = \mathbf{1}$$

(Note for students: We can also write this as $X(s) = \frac{2}{s-2} - \frac{1}{s-3}$ by distributing the operations differently, matching the standard inverse form.)

Take the inverse Laplace transform directly:

$$\boxed{x(t) = 2e^{2t} - e^{3t}}$$

8.3 Example 2

Problem: Solve the system given the initial conditions $x(0) = 0$ and $y(0) = 1$:

$$x' + y = 1$$

$$x - y' = 0$$

8.3.1 Step 1: Transform both equations

$$\mathcal{L}\{x'\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} \implies [sX(s) - x(0)] + Y(s) = \frac{1}{s}$$

$$\mathcal{L}\{x\} - \mathcal{L}\{y'\} = \mathcal{L}\{0\} \implies X(s) - [sY(s) - y(0)] = 0$$

Substitute $x(0) = 0$ and $y(0) = 1$:

$$sX(s) - 0 + Y(s) = \frac{1}{s} \implies sX(s) + Y(s) = \frac{1}{s} \tag{i}$$

$$X(s) - [sY(s) - 1] = 0 \implies X(s) - sY(s) + 1 = 0 \tag{ii}$$

8.3.2 Step 2: Isolate $X(s)$ in equation (ii)

$$X(s) = sY(s) - 1 \tag{iii}$$

8.3.3 Step 3: Substitute equation (iii) into equation (i)

$$s\left[sY(s) - 1\right] + Y(s) = \frac{1}{s}$$

Distribute the outer factor s:

$$s^2Y(s) - s + Y(s) = \frac{1}{s}$$

Group the $Y(s)$ terms together and move the independent variable s to the right side:

$$\left(s^2 + 1\right)Y(s) = \frac{1}{s} + s$$

Find a common denominator for the right side by multiplying s by $\frac{s}{s}$:

$$\left(s^2 + 1\right)Y(s) = \frac{1}{s} + \frac{s^2}{s} \implies \left(s^2 + 1\right)Y(s) = \frac{s^2 + 1}{s}$$

Divide both sides by $\left(s^2 + 1\right)$:

$$Y(s) = \frac{s^2 + 1}{s\left(s^2 + 1\right)}$$

Cancel out the common polynomial term $\left(s^2 + 1\right)$ from both the numerator and denominator:

$$Y(s) = \frac{1}{s}$$

8.3.4 Step 4: Invert to find solutions

Take the inverse transform of $Y(s)$:

$$\boxed{y(t) = 1}$$

Now plug our simplified equation $Y(s) = \frac{1}{s}$ back into equation (iii) to evaluate $X(s)$:

$$X(s) = s\left(\frac{1}{s}\right) - 1 = 1 - 1 = 0$$

Take the inverse transform of 0:

$$\boxed{x(t) = 0}$$

8.4 Example 3

Problem: Solve the system given the initial conditions $x(0) = 1$ and $y(0) = 0$:

$$x' + 2x - y = e^t$$

$$y' + 2y - x = 0$$

8.4.1 Step 1: Transform and Simplify both equations

$$\mathcal{L}\{x'\} + 2\mathcal{L}\{x\} - \mathcal{L}\{y\} = \mathcal{L}\{e^t\} \implies [sX(s) - x(0)] + 2X(s) - Y(s) = \frac{1}{s-1}$$

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} - \mathcal{L}\{x\} = \mathcal{L}\{0\} \implies [sY(s) - y(0)] + 2Y(s) - X(s) = 0$$

Substitute the boundary conditions $x(0) = 1$ and $y(0) = 0$:

$$sX(s) - 1 + 2X(s) - Y(s) = \frac{1}{s-1}$$

$$sY(s) - 0 + 2Y(s) - X(s) = 0$$

Group the terms carefully:

$$(s+2)X(s) - Y(s) = \frac{1}{s-1} + 1 \tag{i}$$

$$(s+2)Y(s) - X(s) = 0 \tag{ii}$$

Simplify the right side of equation (i) by finding a common denominator:

$$\frac{1}{s-1} + \frac{s-1}{s-1} = \frac{1+s-1}{s-1} = \frac{s}{s-1}$$

Our clean simultaneous system is:

$$(s+2)X(s) - Y(s) = \frac{s}{s-1} \tag{i}$$

$$X(s) = (s+2)Y(s) \tag{ii}$$

8.4.2 Step 2: Substitute equation (ii) directly into equation (i)

Replace $X(s)$ with $(s+2)Y(s)$:

$$(s+2)\left[(s+2)Y(s)\right] - Y(s) = \frac{s}{s-1}$$

$$\left[(s+2)^2 - 1\right]Y(s) = \frac{s}{s-1}$$

8.4.3 Step 3: Factor the characteristic expression

Recognize that inside the bracket we have a difference of two squares ($A^2 - B^2 = (A-B)(A+B)$), where $A = (s+2)$ and $B = 1$:

$$[(s+2) - 1] \cdot [(s+2) + 1] = (s+1)(s+3)$$

Substitute this factored form back into the main equation:

$$(s+1)(s+3)Y(s) = \frac{s}{s-1}$$

Divide across by the polynomial blocks to isolate $Y(s)$:

$$Y(s) = \frac{s}{(s-1)(s+1)(s+3)}$$

8.4.4 Step 4: Perform Partial Fraction Decomposition for $Y(s)$

$$\frac{s}{(s-1)(s+1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3}$$

- **Find** A (set $s = 1$):

$$A = \left.\frac{s}{(s+1)(s+3)}\right|_{s=1} = \frac{1}{(1+1)(1+3)} = \frac{1}{(2)(4)} = \mathbf{\frac{1}{8}}$$

- **Find** B (set $s = -1$):

$$B = \left.\frac{s}{(s-1)(s+3)}\right|_{s=-1} = \frac{-1}{(-1-1)(-1+3)} = \frac{-1}{(-2)(2)} = \frac{-1}{-4} = \mathbf{\frac{1}{4}}$$

- **Find** C (set $s = -3$):

$$C = \left.\frac{s}{(s-1)(s+1)}\right|_{s=-3} = \frac{-3}{(-3-1)(-3+1)} = \frac{-3}{(-4)(-2)} = -\mathbf{\frac{3}{8}}$$

Rebuild the final expression components and invert them:

$$Y(s) = \frac{1}{8}\left(\frac{1}{s-1}\right) + \frac{1}{4}\left(\frac{1}{s+1}\right) - \frac{3}{8}\left(\frac{1}{s+3}\right)$$

$$\boxed{y(t) = \frac{1}{8}e^{t} + \frac{1}{4}e^{-t} - \frac{3}{8}e^{-3t}}$$

9 Appendices

9.1 Advanced Laplace Transforms

This table contains specialized Laplace transform pairs for complex combinations of trigonometric, logarithmic, algebraic, and exponential functions often encountered in boundary-value problems and mechanical vibrations.

Time Function $f(t)$ $(t \geq 0)$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$	Constraints / Notes
$\frac{\sin(at)}{t}$	$\arctan\left(\frac{a}{s}\right)$	$s > 0$
$\frac{1-\cos(at)}{t}$	$\frac{1}{2}\ln\left(1+\frac{a^2}{s^2}\right)$	$s > 0$
$\frac{\cos(at)-\cos(bt)}{t}$	$\frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$	$s > 0$
$\frac{e^{bt}-e^{at}}{t}$	$\ln\left(\frac{s-a}{s-b}\right)$	$s > \max(a, b)$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	Amplitude builds over time
$\sin(at) + at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$	Combined resonance effect
$at\sin(at)$	$\frac{2as^2}{(s^2+a^2)^2} - \frac{a}{s^2+a^2} = \frac{2as^2 - a(s^2+a^2)}{(s^2+a^2)^2}$	Often simplifies via common denominator
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	Frequency derivative of $\cos(at)$
$t^2\sin(at)$	$\frac{6as^2-2a^3}{(s^2+a^2)^3}$	High-order mechanical resonance
$t^2\cos(at)$	$\frac{2s^3-6as^2}{(s^2+a^2)^3}$	$s > 0$
$\sin(at+\phi)$	$\frac{s\sin\phi + a\cos\phi}{s^2+a^2}$	Phase-shifted sine wave
$\cos(at+\phi)$	$\frac{s\cos\phi - a\sin\phi}{s^2+a^2}$	Phase-shifted cosine wave
$\ln(t)$	$-\frac{1}{s}[\ln(s)+\gamma]$	$\gamma \approx 0.5772$ (Euler-Mascheroni constant)
$\ln(at)$	$\frac{\ln(a)-\ln(s)-\gamma}{s}$	$s > 0, a > 0$

9.2 Essential Operational Identities Used

To derive or manipulate the advanced combinations above, three structural operational properties are primarily used:

9.2.1 1. Frequency Integration (Division by t)

If $\lim_{t\to 0^+} \dfrac{f(t)}{t}$ exists:

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

9.2.2 2. Frequency Differentiation (Multiplication by t)

$$\mathcal{L}\{t\cdot f(t)\} = -\frac{d}{ds}F(s)$$

$$\mathcal{L}\{t^n\cdot f(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

9.2.3 3. Basic Trigonometric Phase Shift

$$\sin(at+\phi) = \sin(at)\cos(\phi) + \cos(at)\sin(\phi)$$

$$\cos(at+\phi) = \cos(at)\cos(\phi) - \sin(at)\sin(\phi)$$

End of Topic 4 — MAT 102