MAT 102: Ordinary Differential Equations

Topic 5 – Qualitative Theory of Ordinary Differential Equations

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2026-07-06

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Learning Outcomes for Topic 5

By the end of this topic you should be able to:

1. Explain the difference between quantitative and qualitative analysis
2. Identify autonomous systems and find their equilibrium points step by step
3. Use a phase line to classify equilibria of scalar ODEs as stable, unstable, or semi-stable
4. Compute eigenvalues of a 2×2 matrix and use them to classify equilibria of linear systems
5. Find and draw nullclines and use them to describe the flow in the phase plane
6. Apply the full Lotka-Volterra predator-prey analysis: equilibria, Jacobian, eigenvalues, period, and ecological interpretation

1 Introduction — What Is Qualitative Theory and Why Do We Need It?

1.1 The Limitation of "Solving" an ODE

In the previous topics, we have been finding exact formulas for solutions of ODEs. For example, we learned that the solution of $y' = 2y$, $y(0) = 3$ is the exact formula $y(t) = 3e^{2t}$.

This approach — finding a formula — is called **quantitative analysis** because it gives us exact numerical values.

But now imagine you are an ecologist, and you want to model how rabbits and foxes interact in a game reserve. You write down a system of two ODEs. When you try to solve them exactly, you discover that **no formula exists** for the solution. Does that mean you can learn nothing?

No! There is a completely different approach.

Qualitative Analysis — Asking Different Questions

Instead of asking *"What is the exact value of* y at time $t = 5$?", qualitative theory asks:

- **Will the solution grow forever, or will it settle down?**
- **Are there special "resting states" where nothing changes?**
- **If the system is nudged away from a resting state, does it return to it or run away?**
- **Will predator and prey populations oscillate, or will one die out?**

These questions can often be answered **without solving the ODE** — just by analysing the structure of the equation.

Tip

Analogy: Think of a ball on a surface.

- A ball in a bowl will always roll back to the bottom, no matter how you push it –> **stable resting state**
- A ball balanced on top of a hill will roll away if you nudge it even slightly –> **unstable resting state**
- A ball on a flat table stays where you put it –> **neutral (neither stable nor unstable)**

Qualitative theory is about classifying these "resting states" and understanding what happens near them.

2 Autonomous Systems

2.1 What Makes a System "Autonomous"?

Definition 5.1 — Autonomous System

A differential equation (or system) is called **autonomous** if the right-hand side depends only on the dependent variable(s), **not** on the independent variable t (time).

For a single equation: $\frac{dy}{dt} = f(y)$ <– autonomous

For a system of two equations:

$$\frac{dx}{dt} = f(x, y) \qquad \text{and} \qquad \frac{dy}{dt} = g(x, y)$$

The key test: **Does** t appear on the right-hand side?

- **Yes** –> NOT autonomous
- **No** –> autonomous

Why does this matter? For an autonomous system, the "rule of change" is the same at every moment in time. Whether it is noon or midnight, the equations behave the same way. This gives the solutions a very useful geometric structure that we will exploit.

Let us check several equations:

Equation	Autonomous?	Reason
$\frac{dy}{dt} = 3y - y^2$	YES	Only y on the right, no t
$\frac{dy}{dt} = t \sin y$	NO	The factor t appears explicitly
$\frac{dx}{dt} = x - xy$, $\frac{dy}{dt} = -y + xy$	YES	Only x and y on the right
$\frac{dy}{dt} = e^{-t} y$	NO	Factor e^{-t} depends on t
$\frac{dx}{dt} = -x + y^2$, $\frac{dy}{dt} = x + y$	YES	Only x and y appear

2.2 Equilibrium Points — Where the System "Rests"

Definition 5.2 — Equilibrium Point (Critical Point)

A point (x^*, y^*) is called an **equilibrium point** (also: critical point, fixed point, or steady state) of the system if both rates of change are **zero** there:

$$f(x^*, y^*) = 0 \quad \textbf{AND} \quad g(x^*, y^*) = 0$$

What this means: If the system starts exactly at (x^*, y^*), it stays there forever. Nothing changes.

Important: An equilibrium is found by solving the two equations $f = 0$ and $g = 0$ **simultaneously**. This is a system of (possibly nonlinear) algebraic equations.

2.2.1 Step-by-Step Method for Finding Equilibria

Step 1: Set the first equation to zero and find all possibilities for x or y.

Step 2: Set the second equation to zero and find all possibilities for x or y.

Step 3: List all combinations that satisfy **both** equations at the same time.

Example 1: Find all equilibria of $\frac{dx}{dt} = 2x - xy, \quad \frac{dy}{dt} = -y + xy.$

Step 1 — Set first equation to zero:

$$2x - xy = 0$$

Factor out x:

$$x(2 - y) = 0$$

So either $x = 0$ or $y = 2$.

Step 2 — Set second equation to zero:

$$-y + xy = 0$$

Factor out y:

$$y(-1 + x) = 0$$

So either $y = 0$ or $x = 1$.

Step 3 — Find all combinations:

We need to combine one condition from Step 1 with one condition from Step 2:

From Step 1	From Step 2	Point	Valid?
$x = 0$	$y = 0$	$(0, 0)$	YES
$x = 0$	$x = 1$	Contradiction!	NO
$y = 2$	$y = 0$	Contradiction!	NO
$y = 2$	$x = 1$	$(1, 2)$	YES

The equilibria are $(0, 0)$ and $(1, 2)$.

Example 2: Find all equilibria of $\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = x^2 - y$.

Step 1: $x - y = 0 \Rightarrow y = x$

Step 2: $x^2 - y = 0 \Rightarrow y = x^2$

Step 3: We need $y = x$ and $y = x^2$ at the same time. So:

$$x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

- If $x = 0$: $y = 0$ –> Point $(0, 0)$

- If $x = 1$: $y = 1$ –> Point $(1, 1)$

The equilibria are $(0, 0)$ and $(1, 1)$.

Example 3: Find all equilibria of the single scalar ODE $\frac{dy}{dt} = y^2 - 4$.

For a scalar ODE, we just set the right-hand side to zero:

$$y^2 - 4 = 0 \Rightarrow (y - 2)(y + 2) = 0$$

$$y^* = 2 \quad \text{and} \quad y^* = -2$$

There are **two equilibria**: $y = 2$ and $y = -2$.

3 Stability of Equilibria in Scalar ODEs — The Phase Line

Before tackling two-variable systems, let us learn to classify equilibria for a **single** autonomous ODE. This is simpler and builds the intuition we will need.

3.1 The Phase Line

For the scalar ODE $\frac{dy}{dt} = f(y)$:

- Where $f(y) > 0$: y is **increasing** (solution moves right on the number line)
- Where $f(y) < 0$: y is **decreasing** (solution moves left on the number line)
- Where $f(y) = 0$: **equilibrium** (solution does not move)

The **phase line** is just the y-axis with arrows showing the direction of flow.

> **Definition 5.3 — Stability for Scalar ODEs**
>
> At an equilibrium y^*:
>
> - **Asymptotically stable:** Arrows point **toward** y^* from both sides –> solutions near y^*

approach it

- **Unstable:** Arrows point **away from** y^* on both sides –> solutions near y^* escape
- **Semi-stable:** Arrow points toward y^* on one side but away on the other

3.1.1 Step-by-Step Method

Step 1: Find all equilibria (set $f(y) = 0$, solve for y).

Step 2: Choose test points in each interval between equilibria. Evaluate the sign of f at each test point.

Step 3: Draw arrows: $f > 0$ means arrow points right (UP (increasing)), $f < 0$ means arrow points left (DOWN (decreasing)).

Step 4: At each equilibrium, look at the arrows on both sides to classify.

Example 1: Classify equilibria of $\frac{dy}{dt} = y(1-y)$.

Step 1 — Equilibria: $y(1-y) = 0 \Rightarrow y^* = 0$ and $y^* = 1$.

Step 2 — Sign of $f(y) = y(1-y)$: We test one point in each interval.

Interval	Test point	f(test)	Sign	Arrow
$y < 0$	$y = -1$	$(-1)(2) = -2$	$-$	DOWN (decreasing)
$0 < y < 1$	$y = 0.5$	$(0.5)(0.5) = 0.25$	$+$	UP (increasing)
$y > 1$	$y = 2$	$(2)(-1) = -2$	$-$	DOWN (decreasing)

Step 3 — Phase line:

$$\cdots \longleftarrow \underbrace{0}_{\text{equilibrium}} \longrightarrow \underbrace{1}_{\text{equilibrium}} \longleftarrow \cdots$$

(Arrows point **away** from $y = 0$ on the right, and **toward** $y = 1$ from both sides.)

Step 4 — Classification:

- $y^* = 0$: Arrow points **away** on the right side (UP) and also away on the left side (DOWN) — wait, on the left $f < 0$ means y decreases (moves further from 0). And on the right $f > 0$ means y increases (moves toward 1, away from 0). So **both arrows point away from 0** –> **Unstable**.
- $y^* = 1$: On the left $(0 < y < 1)$, $f > 0$ –> y increases –> moves **toward** 1. On the right $(y > 1)$, $f < 0$ –> y decreases –> moves **toward** 1. Both arrows point toward 1 –> **Asymptotically stable**.

Biological meaning: This is the logistic growth model with carrying capacity $y = 1$. Any population starting above zero will eventually approach 1.

Example 2: Classify equilibria of $\dfrac{dy}{dt} = (y-2)(y+1)$.

Step 1 — Equilibria: $(y-2)(y+1) = 0 \Rightarrow y^* = 2$ and $y^* = -1$.

Step 2 — Sign table:

Interval	Test point	$f = (y-2)(y+1)$	Sign	Direction
$y < -1$	$y = -2$	$(-4)(-1) = 4$	$+$	UP (increasing)
$-1 < y < 2$	$y = 0$	$(-2)(1) = -2$	$-$	DOWN (decreasing)
$y > 2$	$y = 3$	$(1)(4) = 4$	$+$	UP (increasing)

Step 3 — Phase line:

$$\cdots \longrightarrow \underbrace{-1}_{\text{stable}} \longleftarrow \underbrace{2}_{\text{unstable}} \longrightarrow \cdots$$

Step 4 — Classification:

- $y^* = -1$: Left arrow points **toward** (UP) and right arrow points **toward** (DOWN). –> **Asymptotically stable**.
- $y^* = 2$: Left arrow points **away** (DOWN) and right arrow points **away** (UP). –> **Unstable**.

Example 3: Classify equilibria of $\frac{dy}{dt} = y^2(y-3)$.

Step 1 — Equilibria: $y^2(y-3) = 0 \Rightarrow y^* = 0$ (double root) and $y^* = 3$.

Step 2 — Sign table:

Interval	Test point	$f = y^2(y-3)$	Sign	Direction
$y < 0$	$y = -1$	$(1)(-4) = -4$	$-$	DOWN
$0 < y < 3$	$y = 1$	$(1)(-2) = -2$	$-$	DOWN
$y > 3$	$y = 4$	$(16)(1) = 16$	$+$	UP

Step 3 — Phase line:

$$\cdots \longleftarrow \underbrace{0}_{\text{semi-stable}} \longleftarrow \underbrace{3}_{\text{unstable}} \longrightarrow \cdots$$

Step 4 — Classification:

- $y^* = 0$: Left arrow points toward 0 (DOWN from left side) but right arrow also points **away** from 3 toward 0 (DOWN toward 0 from the right). Wait — on the right of 0 (i.e., $0 < y < 3$), $f < 0$, so y decreases, meaning it moves **toward** 0 from above. On the left of 0, $f < 0$, y decreases, meaning it moves **away** from 0. So arrows: left side points away, right side points toward –> **Semi-stable** (also called half-stable).
- $y^* = 3$: Left arrow (DOWN) points toward 3, right arrow (UP) points away –> actually left points away from 3 (going downward, toward 0), and right points away (going upward). –> **Unstable**.

4 Stability of Linear Systems — Using Eigenvalues

4.1 What Is a Linear System?

The general **linear autonomous system** in two variables is:

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

where a, b, c, d are constants. This can be written compactly using a **matrix**:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called the **coefficient matrix**.

The **only equilibrium** of this linear system (when $\det A \neq 0$) is the origin $(0, 0)$.

The behaviour of all solutions — whether they flow toward the origin, away from it, or orbit it — is completely determined by the **eigenvalues** of A.

4.2 Background: What Is an Eigenvalue?

Definition 5.4 — Eigenvalues of a 2×2 Matrix

Given the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **eigenvalues** λ satisfy the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

Expanding this determinant:

$$\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

This is a quadratic equation in λ. We write it using:

- **Trace:** $\text{tr}(A) = a + d$ (sum of diagonal entries)
- **Determinant:** $\det(A) = ad - bc$ (product of diagonals minus product of off-diagonals)

So the characteristic equation is:

$$\lambda^2 - \text{tr}(A) \cdot \lambda + \det(A) = 0$$

and the eigenvalues are:

$$\lambda = \frac{\text{tr}(A) \pm \sqrt{[\text{tr}(A)]^2 - 4\det(A)}}{2}$$

In plain words: Eigenvalues are special numbers associated with a matrix. For our purposes, they tell us how solutions behave near the equilibrium.

4.3 How to Find Eigenvalues — Step-by-Step

Step 1: Write down the matrix A.

Step 2: Compute $\text{tr}(A) = a + d$ and $\det(A) = ad - bc$.

Step 3: Write the characteristic equation $\lambda^2 - \text{tr}(A) \cdot \lambda + \det(A) = 0$.

Step 4: Solve the quadratic using factoring or the quadratic formula.

Example: Find the eigenvalues of $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

Step 1: $a = 3, b = 1, c = 2, d = 4$.

Step 2: $\text{tr}(A) = 3 + 4 = 7$. $\det(A) = (3)(4) - (1)(2) = 12 - 2 = 10$.

Step 3: Characteristic equation: $\lambda^2 - 7\lambda + 10 = 0$.

Step 4: Factor: $(\lambda - 2)(\lambda - 5) = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = 5$.

4.4 Classification Table — What the Eigenvalues Tell Us

Once you have the eigenvalues λ_1 and λ_2, use this table:

Type of eigenvalues	Name of equilibrium	Stability	What solutions look like
Both real, both **negative**: $\lambda_2 < \lambda_1 < 0$	**Stable node** (sink/attractor)	Asymptotically stable	Flow curves all approach origin
Both real, both **positive**: $0 < \lambda_1 < \lambda_2$	**Unstable node** (source/repeller)	Unstable	Flow curves all move away from origin
Real, **opposite signs**: $\lambda_1 < 0 < \lambda_2$	**Saddle point**	Unstable	Some curves approach, most move away
Complex: $\lambda = \alpha \pm \beta i$, $\alpha < 0$	**Stable spiral** (spiral sink)	Asymptotically stable	Curves spiral inward (like water down a drain)
Complex: $\lambda = \alpha \pm \beta i$, $\alpha > 0$	**Unstable spiral** (spiral source)	Unstable	Curves spiral outward
Pure imaginary: $\lambda = \pm\beta i$ ($\alpha = 0$)	**Centre**	Lyapunov stable (not asymptotic)	Closed elliptic orbits — perpetual oscillation
Repeated: $\lambda_1 = \lambda_2 = \lambda < 0$	**Stable star node**	Asymptotically stable	Curves approach origin along straight lines
Repeated: $\lambda_1 = \lambda_2 = \lambda > 0$	**Unstable star node**	Unstable	Curves move away

The Golden Rule for Stability

For a **linear system**, the stability of the origin depends only on the **real parts** of the eigenvalues:

$$\text{All } \mathrm{Re}(\lambda) < 0 \quad \Rightarrow \quad \textbf{Asymptotically stable}$$

$$\text{Any } \mathrm{Re}(\lambda) > 0 \quad \Rightarrow \quad \textbf{Unstable}$$

$$\mathrm{Re}(\lambda) = 0 \text{ (pure imaginary)} \quad \Rightarrow \quad \textbf{Centre (stable, not asymptotic)}$$

##Worked Classification Examples

Example 1 — Stable Node:

Classify the equilibrium of $\dfrac{dx}{dt} = -3x + y, \quad \dfrac{dy}{dt} = x - 3y.$

Step 1 — Write the matrix:

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

Step 2 — Compute trace and determinant:

$$\mathrm{tr}(A) = -3 + (-3) = -6, \qquad \det(A) = (-3)(-3) - (1)(1) = 9 - 1 = 8$$

Step 3 — Characteristic equation:

$$\lambda^2 - (-6)\lambda + 8 = 0 \quad \Rightarrow \quad \lambda^2 + 6\lambda + 8 = 0$$

Step 4 — Solve by factoring:

$$(\lambda + 2)(\lambda + 4) = 0 \quad \Rightarrow \quad \lambda_1 = -2, \quad \lambda_2 = -4$$

Step 5 — Classify: Both eigenvalues are real and negative: $-4 < -2 < 0$.

$$\boxed{\text{Stable node — Asymptotically stable}}$$

All solutions flow toward the origin as $t \to \infty$.

Example 2 — Saddle Point:

Classify the equilibrium of $\dfrac{dx}{dt} = x + 2y, \quad \dfrac{dy}{dt} = 3x + 2y.$

Step 1 — Matrix: $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

Step 2: $\text{tr}(A) = 1 + 2 = 3$, $\det(A) = (1)(2) - (2)(3) = 2 - 6 = -4$

Step 3 — Characteristic equation: $\lambda^2 - 3\lambda - 4 = 0$

Step 4 — Solve: $(\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 4$, $\lambda_2 = -1$

Step 5 — Classify: Opposite signs — one positive, one negative.

$$\boxed{\text{Saddle point — Unstable}}$$

Some trajectories flow toward the origin (along the direction of $\lambda_2 = -1$), but most are pushed away (along the direction of $\lambda_1 = 4$).

Example 3 — Stable Spiral:

Classify the equilibrium of $\dfrac{dx}{dt} = -x + 2y, \quad \dfrac{dy}{dt} = -2x - y.$

Step 1 — Matrix: $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

Step 2: $\text{tr}(A) = -1 + (-1) = -2$, $\det(A) = (-1)(-1) - (2)(-2) = 1 + 4 = 5$

Step 3 — Characteristic equation: $\lambda^2 + 2\lambda + 5 = 0$

Step 4 — Solve using the quadratic formula:

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

The eigenvalues are **complex**: $\lambda = -1 \pm 2i$, where the real part $\alpha = -1$ and the imaginary part $\beta = 2$.

Step 5 — Classify: Complex eigenvalues with real part $\alpha = -1 < 0$.

$$\boxed{\text{Stable spiral — Asymptotically stable}}$$

Solutions spiral **inward** toward the origin — like water going down a drain.

Example 4 — Centre:

Classify the equilibrium of $\dfrac{dx}{dt} = 2x - 5y, \quad \dfrac{dy}{dt} = x - 2y.$

Step 1 — Matrix: $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

Step 2: $\text{tr}(A) = 2 + (-2) = 0, \ \det(A) = (2)(-2) - (-5)(1) = -4 + 5 = 1$

Step 3 — Characteristic equation: $\lambda^2 - 0 \cdot \lambda + 1 = 0 \Rightarrow \lambda^2 + 1 = 0$

Step 4 — Solve: $\lambda^2 = -1 \Rightarrow \lambda = \pm i$

Pure imaginary eigenvalues: $\alpha = 0$, $\beta = 1$.

Step 5 — Classify: Real part is exactly zero.

$$\boxed{\text{Centre — Lyapunov stable (not asymptotically stable)}}$$

Solutions travel in **closed ellipses** around the origin forever — perpetual oscillation, no decay.

Example 5 — Unstable Node (Repeated Root):

Classify the equilibrium of $\dfrac{dx}{dt} = 3x + y, \quad \dfrac{dy}{dt} = -x + y.$

Step 1 — Matrix: $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

Step 2: $\text{tr}(A) = 3 + 1 = 4, \ \det(A) = (3)(1) - (1)(-1) = 3 + 1 = 4$

Step 3 — Characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

Step 4 — Solve: $(\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$ (repeated root)

Step 5 — Classify: Repeated positive real eigenvalue.

$$\boxed{\text{Unstable star node — Unstable}}$$

5 The Phase Plane

5.1 What Is the Phase Plane?

When we have a system of two ODEs:

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y)$$

the solution is a **pair of functions** $x(t)$ and $y(t)$. We could plot x vs t and y vs t separately — but this gives us two graphs and we lose the connection between x and y.

A much more powerful approach is the **phase plane**: we plot x on the horizontal axis and y on the vertical axis, and draw the curve $(x(t), y(t))$ as t increases.

Definition 5.5 — Phase Plane, Trajectory, and Phase Portrait

- The **phase plane** is the xy-plane (not the xt or yt plane).
- A **trajectory** (or **orbit**) is the curve traced by the point $(x(t), y(t))$ as t goes from 0 to ∞.
- Each trajectory has a **direction**: an arrow showing which way the point moves as t increases.

- The **phase portrait** is the complete picture of all possible trajectories — it reveals the overall behaviour of all solutions at once.

What does it tell us? The phase portrait shows, for any starting position, where the system eventually goes.

An everyday analogy: Imagine a ship's position (x, y) on a map. The phase portrait is like a weather map showing ocean currents — it tells you where any ship will drift, regardless of where it starts.

5.2 The Direction Field — Arrows at Every Point

At any point (x, y) in the phase plane, the system tells us the **velocity** of the solution:

$$\text{velocity in } x\text{-direction} = f(x, y), \qquad \text{velocity in } y\text{-direction} = g(x, y)$$

So the direction of motion at (x, y) is the vector $(f(x, y),\ g(x, y))$. We can draw a small arrow at every point showing this direction — this gives the **direction field** (or **vector field**).

Trajectories are curves that are always **tangent to these arrows**.

5.3 Nullclines — The Skeleton of the Phase Portrait

Before drawing trajectories, we first draw **nullclines**. These are like the "bones" of the phase portrait — once you have them, you know where the arrows point in every region.

Definition 5.6 — Nullclines

x-nullcline: The set of all points where $\dfrac{dx}{dt} = f(x, y) = 0$.

- On the x-nullcline, the solution has **no horizontal movement** — it can only move **up or down** (vertically).

y-nullcline: The set of all points where $\frac{dy}{dt} = g(x, y) = 0$.

- On the y-nullcline, the solution has **no vertical movement** — it can only move **left or right** (horizontally).

Equilibrium points = intersections of an x-nullcline and a y-nullcline (where both velocities are zero).

5.4 Step-by-Step Phase Portrait Analysis

Here is the complete procedure for analysing a 2×2 autonomous system:

Procedure — Complete Phase Plane Analysis

Step 1. Find all equilibrium points by solving $f(x, y) = 0$ and $g(x, y) = 0$ simultaneously.

Step 2. Find the x-nullclines: set $f(x, y) = 0$ and sketch the resulting curve(s).

Step 3. Find the y-nullclines: set $g(x, y) = 0$ and sketch the resulting curve(s).

Step 4. In each region bounded by nullclines, choose a **test point** and evaluate the signs of f and g to determine the arrow direction:

$f > 0$	$g > 0$	Arrow: $\rightarrow$ and UP = northeast NE
$f > 0$	$g < 0$	Arrow: $\rightarrow$ and DOWN = southeast SE
$f < 0$	$g > 0$	Arrow: $\leftarrow$ and UP = northwest NW
$f < 0$	$g < 0$	Arrow: $\leftarrow$ and DOWN = southwest SW

Step 5. Classify each equilibrium using the eigenvalue method (Section 4).

Step 6. Sketch representative trajectories consistent with the arrows and the equilibrium types.

Example 1 — Full Phase Plane Analysis:

Analyse the system $\frac{dx}{dt} = 2 - x - y, \quad \frac{dy}{dt} = x - y.$

Step 1 — Equilibria:

$f = 0$: $2 - x - y = 0 \Rightarrow y = 2 - x$

$g = 0$: $x - y = 0 \Rightarrow y = x$

Both conditions: $y = 2 - x$ and $y = x$. So $x = 2 - x \Rightarrow 2x = 2 \Rightarrow x = 1$, $y = 1$.

Only equilibrium: $(1, 1)$.

Step 2 — x-nullcline: $y = 2 - x$ (a line with slope -1, y-intercept 2).

Step 3 — y-nullcline: $y = x$ (the diagonal line).

Step 4 — **Direction in each region** (test points):

Test $(0, 0)$: $f = 2 - 0 - 0 = 2 > 0$ (rightward), $g = 0 - 0 = 0$ (on nullcline — choose another point).

Test $(0, 1)$: $f = 2 - 0 - 1 = 1 > 0$, $g = 0 - 1 = -1 < 0$ –> Arrow: SE (right and down).

Test $(2, 0)$: $f = 2 - 2 - 0 = 0$ (on x-nullcline), so try $(2, 0.5)$: $f = 2 - 2 - 0.5 = -0.5 < 0$, $g = 2 - 0.5 = 1.5 > 0$ –> Arrow: NW (left and up).

Step 5 — **Classify equilibrium** $(1, 1)$:

$$A = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}_{(1,1)} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

$\text{tr}(A) = -2$, $\det(A) = 1 + 1 = 2$.

$\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \dfrac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$

Complex eigenvalues with $\alpha = -1 < 0$ –> **Stable spiral — asymptotically stable**.

All trajectories spiral inward toward $(1, 1)$.

Example 2 — Linear System with Full Analysis:

Analyse $\dfrac{dx}{dt} = x - y$, $\quad \dfrac{dy}{dt} = 2x - y$.

Step 1 — **Equilibria:** $x - y = 0 \Rightarrow y = x$. Then $2x - y = 0 \Rightarrow 2x - x = 0 \Rightarrow x = 0$. Only equilibrium: $(0, 0)$.

Step 2 — Nullclines:

x-nullcline ($f = 0$): $y = x$ (the diagonal).

y-nullcline ($g = 0$): $2x - y = 0 \Rightarrow y = 2x$.

Step 3 — Test regions:

Test point $(1, 0)$ (between the two nullclines, above $y = 0$ but below $y = x$... actually below $y = x$ since $0 < 1$):

$f = 1 - 0 = 1 > 0$ (rightward), $g = 2(1) - 0 = 2 > 0$ (upward) –> right + UP (NE).

Test point $(0, 1)$ (above both nullclines):

$f = 0 - 1 = -1 < 0$, $g = 0 - 1 = -1 < 0$ –> left + DOWN (SW).

Step 4 — Eigenvalues:

$A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$, tr $= 0$, det $= -1 + 2 = 1$.

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$ –> **Centre — Lyapunov stable**.

Solutions travel in closed ellipses around the origin.

Example 3 — Identifying Regions with Nullclines:

For $\dfrac{dx}{dt} = x(1 - y), \quad \dfrac{dy}{dt} = y(x - 1)$, identify the direction of flow in all four quadrants of the first quadrant.

Nullclines:

x-nullcline: $x = 0$ (the y-axis) or $y = 1$ (horizontal line).

y-nullcline: $y = 0$ (the x-axis) or $x = 1$ (vertical line).

These create four regions in the first quadrant. Test a point in each:

Region	Test point	$f = x(1 - y)$	$g = y(x - 1)$	Direction
$0 < x < 1$, $0 < y < 1$	$(0.5, 0.5)$	$+$	$-$	SE
$x > 1$, $0 < y < 1$	$(2, 0.5)$	$+$	$+$	NE

Region	Test point	$f = x(1-y)$	$g = y(x-1)$	Direction
$x > 1$, $y > 1$	$(2, 2)$	$-$	$+$	NW
$0 < x < 1$, $y > 1$	$(0.5, 2)$	$-$	$-$	SW

The flow circulates **counterclockwise** around the coexistence equilibrium $(1, 1)$ — suggesting a centre-like behaviour.

6 Application to Predator-Prey Equations — The Lotka-Volterra Model

6.1 Setting the Scene

The **Lotka-Volterra predator-prey model** is one of the most famous mathematical models in biology. It was proposed independently by Alfred Lotka (1925) and Vito Volterra (1926) to explain why fish populations in the Adriatic Sea oscillated even when fishing was stopped.

The biological setup:

- Two species live in the same environment
- **Prey** (e.g., rabbits, fish, wildebeest) have a food supply and reproduce, but are eaten by predators
- **Predators** (e.g., foxes, sharks, lions) need to eat prey to survive and reproduce, but die without food

What do we expect intuitively?

- When prey are plentiful –> predators thrive –> predators eat more prey –> prey decrease
- When prey are scarce –> predators starve –> predator numbers fall –> prey recover
- This cycle repeats –> **oscillations**

6.2 The Lotka-Volterra Equations

> **ℹ The Lotka-Volterra System**
>
> $$\frac{dx}{dt} = ax - bxy \qquad (\textbf{Prey})$$
>
> $$\frac{dy}{dt} = -cy + dxy \qquad (\textbf{Predator})$$
>
> **Variables:**
>
> - $x(t)$ = prey population at time t
> - $y(t)$ = predator population at time t
>
> **Parameters** (all strictly positive):
>
> - a = natural growth rate of prey (in the absence of predators)
> - b = rate at which predators consume prey (predation rate)
> - c = natural death rate of predators (in the absence of prey)
> - d = rate at which predators gain from eating prey (conversion efficiency)

Understanding each term — one by one:

Term	What it represents
$+ax$	Prey grow exponentially on their own (birth rate a per prey per unit time)
$-bxy$	Prey are eaten when predators meet them; xy is proportional to the number of encounters
$-cy$	Predators die exponentially without food (death rate c)
$+dxy$	Predators gain offspring from eating prey

6.3 Step 1 — Finding the Equilibria

Set both equations to zero:

$$ax - bxy = x(a - by) = 0 \quad \Rightarrow \quad x = 0 \;\text{ or }\; y = \frac{a}{b}$$

$$-cy + dxy = y(-c + dx) = 0 \quad \Rightarrow \quad y = 0 \;\text{ or }\; x = \frac{c}{d}$$

Combining:

> **ℹ The Two Equilibria**
>
> **Equilibrium 1 — Extinction:** $(x^*, y^*) = (0, 0)$
>
> Both populations are zero. If we start here, nothing changes. But is it realistic or stable?
>
> **Equilibrium 2 — Coexistence:** $(x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\right)$
>
> Both species persist together. Notice:
>
> - The prey equilibrium $x^* = c/d$ depends on the **predator** parameters (c and d) — not the prey's own parameters!
> - The predator equilibrium $y^* = a/b$ depends on the **prey** parameters (a and b) — not the predator's own parameters!
>
> This is a profound ecological insight — each species' equilibrium is controlled by the other species' properties.

6.4 Step 2 — Classifying the Equilibria Using the Jacobian

For **nonlinear** systems, we cannot use the coefficient matrix A directly. Instead, we **linearise** near each equilibrium using the **Jacobian matrix**.

> **ℹ Definition 5.7 — Jacobian Matrix**
>
> For the system $x' = f(x, y)$, $y' = g(x, y)$, the **Jacobian matrix** at a point (x_0, y_0) is:

$$J(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_0, y_0)}$$

How to compute it:

1. Differentiate $f(x, y)$ with respect to x (treating y as constant) –> put in position (1,1)
2. Differentiate $f(x, y)$ with respect to y (treating x as constant) –> put in position (1,2)
3. Differentiate $g(x, y)$ with respect to x –> position (2,1)
4. Differentiate $g(x, y)$ with respect to y –> position (2,2)
5. Substitute the coordinates of the equilibrium point

The eigenvalues of $J(x_0, y_0)$ classify the equilibrium (x_0, y_0).

For the Lotka-Volterra system, compute the partial derivatives of $f(x, y) = ax - bxy$ and $g(x, y) = -cy + dxy$:

$$\frac{\partial f}{\partial x} = a - by, \quad \frac{\partial f}{\partial y} = -bx, \quad \frac{\partial g}{\partial x} = dy, \quad \frac{\partial g}{\partial y} = -c + dx$$

So the Jacobian is:

$$J(x, y) = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$$

6.4.1 At the Extinction Equilibrium $(0, 0)$:

$$J(0, 0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$$

This is a **diagonal matrix** — the eigenvalues are just the diagonal entries: $\lambda_1 = a > 0$ and $\lambda_2 = -c < 0$.

Eigenvalues have opposite signs –> **Saddle point — Unstable**.

Ecological meaning: If both populations are near zero (but not exactly zero), the system will move away from extinction. Small populations will grow (for prey) or struggle (for predators) — but the equilibrium itself is unstable. This is good news: extinction is an unstable state, so small initial populations can recover.

6.4.2 At the Coexistence Equilibrium $\left(\frac{c}{d}, \frac{a}{b}\right)$:

Substitute $x = c/d$ and $y = a/b$:

$$J\left(\frac{c}{d}, \frac{a}{b}\right) = \begin{pmatrix} a - b \cdot \frac{a}{b} & -b \cdot \frac{c}{d} \\ d \cdot \frac{a}{b} & -c + d \cdot \frac{c}{d} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}$$

Trace: $\text{tr}(J) = 0 + 0 = 0$

Determinant: $\det(J) = (0)(0) - \left(-\frac{bc}{d}\right)\left(\frac{da}{b}\right) = \frac{abcd}{bd} = ac$

Characteristic equation: $\lambda^2 - 0 \cdot \lambda + ac = 0 \Rightarrow \lambda^2 + ac = 0$

$$\lambda = \pm i\sqrt{ac}$$

Pure imaginary eigenvalues (real part = 0) –> **Centre — Lyapunov stable**.

Ecological meaning: Near the coexistence equilibrium, populations oscillate periodically. The period of these oscillations is:

$$\boxed{T = \frac{2\pi}{\sqrt{ac}}}$$

where a is the prey growth rate and c is the predator death rate.

6.5 Step 3 — Nullcline Analysis

x-nullclines (where $dx/dt = 0$): $x = 0$ or $y = a/b$

- $x = 0$: the y-axis (no prey –> prey rate is zero)
- $y = a/b$: a horizontal line

y-nullclines (where $dy/dt = 0$): $y = 0$ or $x = c/d$

- $y = 0$: the x-axis (no predators)
- $x = c/d$: a vertical line

These lines divide the first quadrant into four regions:

Region	dx/dt	dy/dt	Flow direction
$x < c/d$, $y < a/b$	$+$	$-$	right and DOWN
$x > c/d$, $y < a/b$	$+$	$+$	right and UP
$x > c/d$, $y > a/b$	$-$	$+$	left and UP
$x < c/d$, $y > a/b$	$-$	$-$	left and DOWN

The flow is counterclockwise around the coexistence equilibrium — consistent with closed orbit (centre) behaviour.

6.6 Step 4 — The Conservation Law (Why Orbits Are Closed)

For most systems with a centre, the linearisation gives only an approximation. For the Lotka-Volterra system, we can prove **exactly** that orbits are closed by finding a **conservation quantity** — a function $V(x, y)$ that is constant along every trajectory.

❗ Theorem — Conservation Law for Lotka-Volterra

The function

$$V(x, y) = dx - c \ln x + by - a \ln y$$

is **constant along every trajectory** of the Lotka-Volterra system. That is:

$$\frac{dV}{dt} = 0 \quad \text{for all solutions}$$

Because V is conserved, every trajectory lies on a level curve $V(x, y) = k$ (constant), and these level curves are **closed loops** around the coexistence equilibrium.

Verification (shown step by step):

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt}$$

$$= \left(d - \frac{c}{x}\right)(ax - bxy) + \left(b - \frac{a}{y}\right)(-cy + dxy)$$

Expand the first product:

$$= d(ax - bxy) - \frac{c}{x}(ax - bxy) = adx - bdxy - ac + bcy$$

Expand the second product:

$$= b(-cy + dxy) - \frac{a}{y}(-cy + dxy) = -bcy + bdxy + ac - adx$$

Add everything together:

$$\frac{dV}{dt} = adx - bdxy - ac + bcy - bcy + bdxy + ac - adx = 0 \checkmark$$

Every term cancels. The conservation law is exact.

6.7 Step 5 — The Ecological Cycle Explained

The closed orbits in the phase plane correspond to **periodic oscillations in time**. Here is the cycle:

> The Predator-Prey Oscillation Cycle
>
> **The cycle of oscillation:**
>
> 1. Prey are plentiful –> predators eat well –> predator numbers rise
> 2. Predators are numerous –> prey are depleted –> prey numbers fall
>
> 3. Prey are scarce –> predators starve –> predator numbers fall
> 4. Predators are few –> prey recover –> prey numbers rise
> 5. Return to step 1 – the cycle repeats
>
> **Key feature:** The predator population peaks *after* the prey population peaks – there is a **time lag** between the two oscillations.
>
> **Key feature:** The predator population **peaks after** the prey population peaks — there is a **time lag** between the two oscillations.

The period of oscillation (from the imaginary part of the eigenvalues):

$$T = \frac{2\pi}{\sqrt{ac}}$$

A larger product ac means a **shorter period** — the oscillations are faster. Increasing the prey growth rate a or the predator death rate c speeds up the cycle.

6.8 A Complete Numerical Example

Example: Consider the predator-prey system:

$$\frac{dx}{dt} = 2x - xy, \qquad \frac{dy}{dt} = -y + 0.5\,xy$$

Parameters: $a = 2$, $b = 1$, $c = 1$, $d = 0.5$.

Step 1 — Find equilibria:

$$x^* = \frac{c}{d} = \frac{1}{0.5} = 2, \qquad y^* = \frac{a}{b} = \frac{2}{1} = 2$$

Equilibria: $(0, 0)$ and $(2, 2)$.

Step 2 — Compute the Jacobian:

$$J(x, y) = \begin{pmatrix} 2 - y & -x \\ 0.5y & -1 + 0.5x \end{pmatrix}$$

At $(0, 0)$:

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2 > 0$, $\lambda_2 = -1 < 0$ –> **Saddle — Unstable** [check]

At $(2, 2)$:

$$J(2, 2) = \begin{pmatrix} 2 - 2 & -2 \\ 0.5 \cdot 2 & -1 + 0.5 \cdot 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

$\text{tr} = 0$, $\det = 0 \cdot 0 - (-2)(1) = 2$

$\lambda^2 + 2 = 0 \Rightarrow \lambda = \pm i\sqrt{2}$ –> **Centre — Lyapunov stable** [check]

Step 3 — Period of oscillations:

$$T = \frac{2\pi}{\sqrt{ac}} = \frac{2\pi}{\sqrt{2 \times 1}} = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2} \approx 4.44 \text{ time units}$$

Step 4 — Conservation quantity:

$$V(x, y) = 0.5\,x - \ln x + y - 2\ln y = \text{constant}$$

Step 5 — Nullcline analysis:

x-nullclines: $x = 0$ or $y = 2$.

y-nullclines: $y = 0$ or $x = 2$.

Test point $(1, 1)$ (region: $x < 2$, $y < 2$):

$dx/dt = 2(1) - (1)(1) = 1 > 0$ –> rightward.

$dy/dt = -(1) + 0.5(1)(1) = -0.5 < 0$ –> downward.

Direction: right + DOWN (SE). Consistent with counterclockwise orbit. [check]

Step 6 — Ecological conclusions:

- The two species coexist in balance at prey = 2, predators = 2.
- If disturbed, both populations oscillate with period approximately 4.44 time units.
- The prey population peak is followed by a predator peak.
- Neither species goes extinct (since $(0, 0)$ is unstable).

6.9 Second Numerical Example

Example: Wildebeest and lions in a game reserve:

$$\frac{dx}{dt} = 0.4x - 0.01xy, \qquad \frac{dy}{dt} = -0.2y + 0.002xy$$

Parameters: $a = 0.4$, $b = 0.01$, $c = 0.2$, $d = 0.002$.

Equilibrium populations:

$$x^* = \frac{c}{d} = \frac{0.2}{0.002} = 100 \text{ wildebeest}, \qquad y^* = \frac{a}{b} = \frac{0.4}{0.01} = 40 \text{ lions}$$

Period of oscillation:

$$T = \frac{2\pi}{\sqrt{ac}} = \frac{2\pi}{\sqrt{0.4 \times 0.2}} = \frac{2\pi}{\sqrt{0.08}} = \frac{2\pi}{0.283} \approx 22.2 \text{ years}$$

Ecological conclusions:

- At balance: 100 wildebeest and 40 lions coexist.
- Population cycles take approximately 22 years to complete.
- When wildebeest exceed 100, lion numbers rise; when wildebeest fall below 100, lion numbers decline.
- Lion peaks follow wildebeest peaks by several years.

6.10 Third Numerical Example

Example: Classify and analyse $\frac{dx}{dt} = 3x - 0.5xy, \quad \frac{dy}{dt} = -2y + 0.25xy.$

Parameters: $a = 3$, $b = 0.5$, $c = 2$, $d = 0.25$.

Equilibria:

$$x^* = \frac{c}{d} = \frac{2}{0.25} = 8, \qquad y^* = \frac{a}{b} = \frac{3}{0.5} = 6$$

Equilibria: $(0, 0)$ and $(8, 6)$.

Jacobian:

$$J(x, y) = \begin{pmatrix} 3 - 0.5y & -0.5x \\ 0.25y & -2 + 0.25x \end{pmatrix}$$

At $(0, 0)$: $J = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, eigenvalues 3 and -2 –> **Saddle — Unstable**.

At $(8, 6)$: $J = \begin{pmatrix} 0 & -4 \\ 1.5 & 0 \end{pmatrix}$

$\text{tr} = 0$, $\det = 6$, $\lambda = \pm i\sqrt{6}$ –> **Centre — Lyapunov stable**.

Period:

$$T = \frac{2\pi}{\sqrt{3 \times 2}} = \frac{2\pi}{\sqrt{6}} \approx 2.57 \text{ time units}$$

Conservation law: $V(x, y) = 0.25x - 2\ln x + 0.5y - 3\ln y = C$