Topic 5 – Qualitative Theory of Ordinary Differential Equations

Tutorial Questions

Dr. Anna Fome | Jordan University College

2026-07-06

Table of contents

> **!** Instructions
>
> Show all working. For every phase plane question, you must: (i) Find all equilibria step by step (ii) Compute eigenvalues (show the characteristic equation) (iii) Classify and state the stability (iv) Describe what solutions look like near the equilibrium

1 Section A: Autonomous Systems and Equilibria

Q1. State whether each ODE or system is autonomous. Give a reason.

(a) $\dfrac{dy}{dt} = y^2 - 5y + 6$

(b) $\dfrac{dy}{dt} = e^{-t}\cos y$

(c) $\dfrac{dx}{dt} = x - 2y + 3,\ \dfrac{dy}{dt} = xy - y^2$

(d) $\dfrac{dx}{dt} = tx + y,\ \dfrac{dy}{dt} = x - ty$

Q2. Find all equilibrium points of each system. Show all steps clearly.

(a) $\dfrac{dx}{dt} = 2x - 4, \quad \dfrac{dy}{dt} = 3y + 6$

(b) $\dfrac{dx}{dt} = x(x - 2), \quad \dfrac{dy}{dt} = y(y - 3)$

(c) $\dfrac{dx}{dt} = x - xy, \quad \dfrac{dy}{dt} = -2y + xy$

(d) $\dfrac{dx}{dt} = x^2 - y, \quad \dfrac{dy}{dt} = y - x$

(e) $\dfrac{dx}{dt} = 4x - 2xy, \quad \dfrac{dy}{dt} = -y + 0.5xy$

2 Section B: Phase Line Analysis for Scalar ODEs

Q3. For each scalar ODE, (i) find all equilibria, (ii) complete a sign table for $f(y)$, (iii) draw the phase line with arrows, (iv) classify each equilibrium.

(a) $\dfrac{dy}{dt} = y(4 - y)$

(b) $\dfrac{dy}{dt} = y^2 - 9$

(c) $\dfrac{dy}{dt} = (y - 1)(y - 3)(y + 2)$

(d) $\dfrac{dy}{dt} = y^2(y - 2)$

3 Section C: Eigenvalues and Classification of Linear Systems

Q4. For each matrix, compute the trace, determinant, characteristic equation, eigenvalues, and classify the origin. State the stability.

(a) $A = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 5 & 3 \\ -2 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} -1 & 4 \\ -1 & -1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

(e) $A = \begin{pmatrix} 3 & -9 \\ 1 & -3 \end{pmatrix}$

Q5. For each linear system, write the matrix A, find eigenvalues, classify the equilibrium at the origin, and describe the long-term behaviour of all solutions.

(a) $x' = -5x + 2y, \quad y' = x - 4y$

(b) $x' = 2x + 3y, \quad y' = -x + 4y$

(c) $x' = x - 5y, \quad y' = 2x - y$

(d) $x' = -2x - y, \quad y' = x - 2y$

(e) $x' = -3x, \quad y' = -5y$ *(diagonal system — what are the eigenvalues?)*

4 Section D: Jacobian and Nonlinear Systems

Q6. For each system, (i) find all equilibria, (ii) compute the Jacobian matrix $J(x, y)$, (iii) evaluate J at each equilibrium, (iv) find eigenvalues and classify.

(a) $\dfrac{dx}{dt} = x - x^2 - xy, \quad \dfrac{dy}{dt} = -y + xy$

(b) $\dfrac{dx}{dt} = 1 - xy, \quad \dfrac{dy}{dt} = x - y^3$

(c) $\dfrac{dx}{dt} = x(2 - x - y), \quad \dfrac{dy}{dt} = y(1 - y + x)$ *(competition model)*

5 Section E: Phase Plane and Nullclines

Q7. For each system, (i) find and sketch the x-nullclines and y-nullclines on a rough diagram, (ii) mark all equilibria, (iii) determine the direction of flow (arrow) in each region using test points.

(a) $x' = x - 2, \quad y' = y - 1$

(b) $x' = x(1 - x - y), \quad y' = y(0.5 - y + 0.5x)$

(c) $x' = y, \quad y' = -x$

Q8. For the system $x' = -y$ and $y' = x$:

(a) Find the eigenvalues of the coefficient matrix.

(b) Classify the equilibrium and state its stability type.

(c) Solve the IVP with $x(0) = 2$, $y(0) = 0$.

(d) Show that $x(t)^2+y(t)^2$ is constant — what does this tell you about the shape of the trajectories?

6 Section F: Predator-Prey Models

Q9. A predator-prey model is given by:

$$\frac{dx}{dt} = 3x - 2xy, \qquad \frac{dy}{dt} = -y + xy$$

(a) Identify the parameters a, b, c, d.

(b) Find both equilibria.

(c) Write the Jacobian matrix $J(x, y)$.

(d) Evaluate J at the extinction equilibrium and find its eigenvalues. Classify and explain ecologically.

(e) Evaluate J at the coexistence equilibrium and find its eigenvalues. Classify and explain ecologically.

(f) Find the period of population oscillations near the coexistence equilibrium.

(g) Write the conservation quantity $V(x, y)$.

Q10. A lake contains trout (prey, x) and pike (predators, y) governed by:

$$\frac{dx}{dt} = 0.6x - 0.03xy, \qquad \frac{dy}{dt} = -0.4y + 0.01xy$$

(a) Find the coexistence equilibrium (the long-term average populations).

(b) Find the approximate period of oscillation.

(c) A fishing ban doubles the trout population overnight (so x is doubled but y stays the same). Starting from this new point, describe qualitatively what happens over the following years.

(d) Compute the Jacobian at the coexistence equilibrium and find the eigenvalues.

Q11. Consider the general Lotka-Volterra system with parameters $a, b, c, d > 0$.

(a) Show algebraically that the Jacobian at $(0, 0)$ always gives a **saddle point**.

(b) Show algebraically that the Jacobian at $(c/d,\ a/b)$ always gives **pure imaginary eigenvalues** $\lambda = \pm i\sqrt{ac}$.

(c) Using part (b), write down the period T of oscillations in terms of a and c.

(d) If we double the prey growth rate a (perhaps due to improved food supply), what happens to the period? Explain in ecological terms.