MAT 103: Numerical Analysis I

Topic 1: Basic Concepts

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> *"All models are wrong, but some are useful."* — George Box, Statistician

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1 Introduction

Have you ever wondered how engineers calculate the exact load a bridge can carry before it breaks? Or how a weather forecast is computed? Or how GPS devices determine your position to within a few metres?

These problems all involve mathematics — but not the kind where you can simply write down a formula and read off the answer. They involve **equations that are too complex to solve exactly**, functions that behave unpredictably, and systems with thousands of interacting variables.

This is precisely where **Numerical Analysis** comes in.

Numerical Analysis is the branch of mathematics that develops and studies methods for solving mathematical problems **approximately**, using arithmetic operations that a computer can carry out efficiently.

> **i What is Numerical Analysis?**
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> **Numerical Analysis** is the study of algorithms (step-by-step procedures) for obtaining approximate solutions to mathematical problems using finite arithmetic operations.

In this course, you will learn to:

- Understand *why* we sometimes cannot solve problems exactly.
- Derive and apply methods that give *good enough* approximate solutions.

- Measure and control the errors in those approximations.
- Use software tools (MATLAB/MAPLE) to implement these methods efficiently.

By the end of Topic 1, you should be able to:

- Distinguish between analytical and numerical methods.
- Explain why numerical methods are necessary.
- Distinguish between direct and iterative methods.
- State the desirable properties of a good numerical method.

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2 Analytical Methods

2.1 What is an Analytical Method?

An **analytical method** (also called an *exact method* or *closed-form method*) is a procedure that produces an **exact solution** to a mathematical problem using a finite sequence of well-known operations: algebra, calculus, known formulas, and logical reasoning.

When we say a solution is *exact*, we mean it can be expressed precisely using mathematical symbols — fractions, roots, trigonometric functions, logarithms, and so on — without any approximation.

2.2 Key Characteristics of Analytical Methods

- Produce **exact** solutions.
- Based on established **mathematical theory**.
- Express answers in **closed form** (a formula).
- Work best for **simple or idealized** problems.

2.3 Solved Examples

Example 1.1: Quadratic Equation

Solve $x^2 - 5x + 6 = 0$.

Solution:

We use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 1$, $b = -5$, $c = 6$. Substituting:

$$x = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2}$$

Therefore $x = 3$ or $x = 2$. These are **exact** values.

Example 1.2: Linear Equation

Solve $3x + 7 = 0$.

Solution:

$$3x = -7 \implies x = -\frac{7}{3}$$

The answer is exact — even though it is a fraction, it is not an approximation.

Example 1.3: Definite Integral

Evaluate $\int_0^1 (3x^2 + 2x)\,dx$.

Solution:

$$\int_0^1 (3x^2 + 2x)\,dx = \left[x^3 + x^2\right]_0^1 = (1 + 1) - (0 + 0) = 2$$

Exact answer: 2.

Example 1.4: System of Two Equations

Solve:

$$2x + y = 7$$
$$x - y = 2$$

Solution:

Adding the two equations: $3x = 9$, so $x = 3$.

Substituting back: $y = 7 - 2(3) = 1$.

Exact solution: $x = 3$, $y = 1$.

Example 1.5: Ordinary Differential Equation

Solve $\frac{dy}{dx} = 3x^2$, with $y(0) = 1$.

Solution:

Integrating both sides:

$$y = \int 3x^2 \, dx = x^3 + C$$

Applying $y(0) = 1$: $C = 1$.

Exact solution: $y = x^3 + 1$.

Think About It

In each example above, we got a neat, exact answer. Can you think of a situation in your everyday life where an exact answer is not possible? Discuss with your neighbour.

2.4 Limitations of Analytical Methods

As powerful as analytical methods are, they come with important limitations. Understanding these limitations is what motivates the need for numerical methods.

Limitation 1: Non-existence of a Closed-Form Solution

Some equations simply cannot be solved using any combination of elementary functions. These are called **transcendental equations**.

Example: Find x such that $e^x = 3x + 1$.

No algebraic manipulation — no matter how clever — can isolate x here. There is no formula for the answer. Yet the equation clearly has a solution (you can sketch the graphs of $y = e^x$ and $y = 3x + 1$ and see where they cross). We *must* use a numerical method.

Limitation 2: Mathematical Complexity

Sometimes a closed-form solution exists in theory, but it is so complicated that it is practically useless for computation.

Example: The general formula for roots of a degree-5 (or higher) polynomial involves expressions so complex they are rarely used in practice. For a degree-5 polynomial, no general algebraic formula exists at all (Abel–Ruffini theorem).

Limitation 3: Data-Defined Functions

In many real-world problems, the function $f(x)$ is not given by a formula — it is known only through **experimental measurements** or **sensor readings**.

Example: A temperature sensor records T at times $t = 0, 1, 2, 3, \dots$ seconds. There is no formula for $T(t)$ — only the data. Yet we may need to compute $\int_0^{10} T(t)\,dt$ (total heat). We must use numerical integration.

Limitation 4: Real-World Complexity

Most real-world systems are governed by **systems of nonlinear differential equations** with many interacting variables. Analytical solutions to such systems rarely exist.

Example: Modelling the spread of a disease in a population (SIR model) leads to a system of nonlinear ODEs with no closed-form solution. Doctors and policymakers still need predictions — so numerical methods are used.

Limitation 5: Simplifying Assumptions

To obtain analytical solutions, we often *simplify* the problem (e.g., assume a beam is perfectly straight, ignore air resistance, linearize a nonlinear system). These simplifications may make the model inaccurate.

> ⚠ Important Observation
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> The fact that analytical methods fail in these situations does **not** mean mathematics has failed. It means we need a different set of tools — and that is exactly what numerical methods provide.

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3 Numerical Methods

3.1 What is a Numerical Method?

A **numerical method** is a step-by-step computational procedure (an algorithm) that produces an **approximate solution** to a mathematical problem. The approximation is close enough to the true answer to be useful in practice.

The key word is *approximate*. Numerical methods do not give exact answers — but they give answers that are as accurate as we need them to be.

> ℹ Definition
>
> A **numerical method** is an algorithm that uses a finite sequence of arithmetic operations (addition, subtraction, multiplication, division) to produce an approximate solution to a mathematical problem.

3.2 A Motivating Example

Problem: Find the solution of $e^x = 3x + 1$.

Step 1: Rewrite as $f(x) = 0$.

Let $f(x) = e^x - 3x - 1$. We want x such that $f(x) = 0$.

Step 2: Look for a sign change.

x	$f(x) = e^x - 3x - 1$
-1	$e^{-1} - 3(-1) - 1 = 0.368 + 3 - 1 = 2.368 > 0$
0	$e^0 - 0 - 1 = 0$
1	$e^1 - 3 - 1 = 2.718 - 4 = -1.282 < 0$
2	$e^2 - 6 - 1 = 7.389 - 7 = 0.389 > 0$

We found $f(0) = 0$ exactly — so $x = 0$ is one root!

There is also a root between $x = 1$ and $x = 2$ (sign change). Narrowing down:

x	$f(x)$
1.5	$e^{1.5} - 5.5 = 4.482 - 5.5 = -1.018 < 0$
1.9	$e^{1.9} - 6.7 = 6.686 - 6.7 = -0.014 < 0$
1.95	$e^{1.95} - 6.85 = 7.029 - 6.85 = 0.179 > 0$
1.91	$e^{1.91} - 6.73 = 6.753 - 6.73 = 0.023 > 0$

So the second root is approximately $x \approx 1.91$.

Notice

We did not use any fancy formula. We used simple arithmetic to *home in* on the answer. This is the essence of numerical methods — systematic, arithmetic-based refinement.

3.3 Key Characteristics of Numerical Methods

- Produce **approximate** solutions.
- Use **step-by-step (iterative)** procedures.
- Applicable to **complex and real-world** problems.
- Require **error analysis** to assess accuracy.
- Typically implemented on a **computer**.

3.4 Brief Overview of Common Numerical Methods

You will study these in detail in later topics. Here is a preview:

Method	Purpose
Bisection Method	Find roots of $f(x) = 0$ by halving an interval
Newton–Raphson Method	Find roots using tangent lines (fast convergence)
Secant Method	Like Newton–Raphson but without needing derivatives
Gaussian Elimination	Solve systems of linear equations
Trapezoidal / Simpson's Rules	Approximate definite integrals
Lagrange Interpolation	Fit a polynomial through data points

3.5 Limitations of Numerical Methods

Numerical methods are powerful but not perfect. Their main limitations include:

1. **Approximation Errors.** Every numerical answer carries some error. Measuring and controlling this error is a central concern of the course.
2. **Convergence Issues.** Some iterative methods may fail to converge, or converge very slowly. The choice of starting value and method matters greatly.
3. **Sensitivity to Initial Conditions.** A poor starting guess can lead to the wrong answer or to divergence.
4. **Computational Cost.** Highly accurate approximations may require many iterations and significant computer time.
5. **Stability.** Some methods amplify errors as computation proceeds, making results unreliable (we study this in Topic 2).
6. **No General Formula.** A numerical answer is a specific number, not a formula. It gives no insight into how the solution depends on the parameters of the problem.

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4 Analytical vs Numerical Methods: A Comparison

Feature	Analytical Methods	Numerical Methods
Type of answer	Exact (closed-form)	Approximate

Feature	Analytical Methods	Numerical Methods
Applicability	Limited (simple problems)	Very broad (complex problems)
Requires formula	Yes	No
Error introduced	None	Yes (must be measured)
Computer needed	Usually not	Yes, for large problems
Works with data tables	No	Yes
Works with nonlinear systems	Rarely	Yes

Key Insight

Analytical and numerical methods are **complementary**, not competitors. When an analytical solution exists and is practical, use it. When it does not — or is too complex — numerical methods step in.

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5 Direct vs Iterative Methods

Numerical methods can be broadly classified as either **direct** or **iterative**.

5.1 Direct Methods

A **direct method** produces the solution in a **finite, predetermined number of arithmetic operations**. In the absence of rounding errors, a direct method gives the exact answer.

Think of it like following a recipe: you perform exactly the same steps every time, and after a fixed number of steps, you have your answer.

5.1.1 Key Characteristics

- Finite number of steps, fixed in advance.
- Exact in theory (errors arise only from rounding).
- Best for **small to medium-sized** problems.
- Memory-intensive for large problems.

5.1.2 Solved Example: Gaussian Elimination (Direct Method)

Solve the system:

$$x + 2y = 5 \quad \cdots (1)$$
$$3x + 4y = 11 \quad \cdots (2)$$

Step 1: Eliminate x from equation (2).

Multiply equation (1) by 3: $3x + 6y = 15$.

Subtract from equation (2): $(3x + 4y) - (3x + 6y) = 11 - 15$

$$-2y = -4 \implies y = 2$$

Step 2: Back-substitute into equation (1).

$$x + 2(2) = 5 \implies x = 1$$

Solution: $x = 1$, $y = 2$.

This took exactly **2 steps** — the same 2 steps every time, regardless of the right-hand side. That makes it a **direct method**.

5.2 Iterative Methods

An **iterative method** starts from an **initial guess** and repeatedly applies a formula to produce a sequence of approximations:

$$x_0 \to x_1 \to x_2 \to x_3 \to \cdots$$

Each new approximation is (ideally) closer to the true answer than the previous one. We stop when the approximation is *close enough* — i.e., when the change between successive steps is smaller than a chosen tolerance ε.

Think of it like adjusting the focus on a camera: you turn the dial repeatedly, each time getting a clearer image, until the picture is sharp enough.

5.2.1 Key Characteristics

- Number of steps **not known in advance**.
- Stop when a **convergence criterion** is met: $|x_{n+1} - x_n| < \varepsilon$.
- Do not require storing large matrices.
- Best for **large or sparse** systems, and for root-finding.

5.2.2 Solved Example: Fixed-Point Iteration

Solve $x^2 - x - 2 = 0$ using the rearrangement $x = \sqrt{x + 2}$, starting from $x_0 = 2$.

The iteration formula is: $x_{n+1} = \sqrt{x_n + 2}$

Step-by-step:

n	x_n	$x_{n+1} = \sqrt{x_n + 2}$	$\|x_{n+1} - x_n\|$
0	2.0000	$\sqrt{4.0000} = 2.0000$	—
1	2.0000	$\sqrt{4.0000} = 2.0000$	0.0000

We see immediately: $x_1 = \sqrt{2+2} = \sqrt{4} = 2 = x_0$. The iteration has converged in one step because $x = 2$ is a fixed point: $\sqrt{2+2} = 2$.

Let us try a less obvious case. Solve $\cos(x) = x$, starting from $x_0 = 1.0$ (angles in radians).

Iteration: $x_{n+1} = \cos(x_n)$

n	x_n	$x_{n+1} = \cos(x_n)$	$\|x_{n+1} - x_n\|$
0	1.0000	0.5403	—
1	0.5403	0.8576	0.3173
2	0.8576	0.6543	0.2033
3	0.6543	0.7935	0.1392
4	0.7935	0.7014	0.0921
5	0.7014	0.7640	0.0626
10	0.7390	0.7391	0.0001

After about 10 iterations, $x \approx 0.7391$. This is the **Dottie number**, the unique fixed point of the cosine function.

Think About It

Why did we stop at 10 iterations? What would happen if we kept going? How do we decide when to stop? This is the concept of a **stopping criterion**, which we will study in Topic 3.

5.3 Direct vs Iterative Methods: Comparison

Feature	Direct Methods	Iterative Methods
Number of steps	Fixed (finite)	Variable
Exact in theory?	Yes	Only in the limit
Memory usage	High (stores whole matrix)	Low
Best suited for	Small/dense systems	Large/sparse systems
Sensitivity to rounding	Can accumulate	Can be controlled

Feature	Direct Methods	Iterative Methods
Examples	Gaussian elimination, Cramer's rule	Newton-Raphson, Bisection, Jacobi

> **i** Real-World Analogy
>
> **Direct method:** Solving a Sudoku puzzle by systematically filling every cell according to fixed logical rules — you take a fixed number of steps.
> **Iterative method:** Adjusting a recipe by taste — you add a little salt, taste it, add a little more, taste again, until it is just right. The number of adjustments depends on how close you started.

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6 Desirable Properties of Numerical Methods

Not every numerical method is equally good. A good numerical method should possess the following properties:

6.1 For Direct Methods

6.1.1 1. Accuracy

A method is **accurate** if the computed solution is close to the true solution. We measure accuracy using the **error**:

$$\text{Error} = |x_{\text{true}} - x_{\text{computed}}|$$

A smaller error means higher accuracy.

Example: If the true answer is $\pi = 3.14159\ldots$ and our method gives 3.14, the error is:

$$|3.14159 - 3.14000| = 0.00159$$

6.1.2 2. Stability

A method is **stable** if small errors (such as rounding errors in intermediate steps) do not grow and destroy the final answer.

Imagine balancing a ball on top of a hill versus inside a bowl:

- On top of a hill: a tiny push sends the ball rolling away — **unstable**.
- Inside a bowl: a tiny push causes the ball to oscillate and settle back — **stable**.

A stable numerical method is like the ball in a bowl — small errors stay small.

An **unstable** method amplifies errors at each step, eventually giving a completely wrong answer.

6.2 For Iterative Methods

6.2.1 3. Convergence

An iterative method **converges** if the sequence of approximations $x_0, x_1, x_2, \ldots$ gets closer and closer to the true answer x^*:

$$\lim_{n \to \infty} x_n = x^*$$

If a method does not converge — if the iterates wander or diverge — it is useless.

Example: In the $\cos(x) = x$ example above, the iterates $1.0, 0.5403, 0.8576, \ldots$ were converging to 0.7391. This is a **convergent** iterative method.

Rate of convergence matters too: a method that converges in 5 steps is better than one that needs 500 steps for the same accuracy.

6.3 Summary of Desirable Properties

Property	Applies To	Meaning
Accuracy	Direct methods	Computed answer is close to the true answer
Stability	Direct methods	Small errors do not grow during computation
Convergence	Iterative methods	Iterates approach the true answer

⚠ A method can be accurate but unstable, or stable but slow to converge.

The art of numerical analysis is finding methods that have **all** the desirable properties simultaneously, and understanding the trade-offs when they cannot all be achieved.

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7 Why Numerical Methods? — Illustrative Examples

Here we present real-world situations where analytical methods fail and numerical methods are essential. These examples are meant to show you the *relevance* of what you are studying.

7.1 Example A: Finding Where Two Curves Cross

Problem: A biologist models two competing species. Species 1 grows as $P_1(t) = e^{0.3t}$ and species 2 grows as $P_2(t) = 1 + 2t + 0.5t^2$. At what time t are their populations equal?

Set $P_1(t) = P_2(t)$:

$$e^{0.3t} = 1 + 2t + 0.5t^2$$

This is a transcendental equation — **no closed-form solution**. A numerical method gives $t \approx 5.38$ years.

7.2 Example B: Engineering Design

Problem: A civil engineer needs to find the cross-sectional area of an irregularly shaped beam. The boundary of the cross-section is measured at 7 points:

x (cm)	0	1	2	3	4	5	6
y (cm)	0	2.1	3.5	4.0	3.7	2.8	0

There is no formula for $y(x)$. To find the area $A = \int_0^6 y(x)\,dx$, the engineer uses **numerical integration** (Topic 5): $A \approx 17.65$ cm^2.

7.3 Example C: Solving a Large System

Problem: A structural engineer models a bridge as 1000 interconnected beams. The forces in the beams satisfy a system of 1000 linear equations $Ax = b$.

Solving 1000 equations by hand using Cramer's rule would require computing 1001 determinants of 1000×1000 matrices — effectively impossible.

Gaussian elimination (Topic 6) solves this in about $\frac{2}{3}(1000)^3 \approx 667$ million arithmetic operations — a matter of seconds on a modern computer.

Reflection

Look around you. Every building, bridge, machine, smartphone, and medical device you see was designed using numerical methods at some stage. You are learning the mathematics that makes the modern world work.

8 Topic Summary

Let us consolidate what we have covered in Topic 1.

Concept	Key Idea
Analytical Method	Produces an exact, closed-form solution
Numerical Method	Produces an approximate solution using arithmetic
Why Numerical?	No formula exists; data given as tables; problem too complex
Direct Method	Exact solution in a fixed number of steps (e.g., Gaussian elimination)
Iterative Method	Converging sequence of approximations (e.g., fixed-point iteration)
Accuracy	Computed answer is close to the true answer
Stability	Small errors do not grow during computation
Convergence	Iterates approach the true solution

i Looking Ahead

In **Topic 2**, we will study errors in detail — how they arise, how they are measured, and how they can be controlled. This is the foundation of everything else in the course.

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9 Tutorial Questions

Work through these questions carefully. Show all working clearly. Questions marked with (*) are more challenging.

9.1 Section A: Short Answer Questions

Question 1

Define the term *numerical analysis* in your own words. Why is it important in modern science and engineering? Give **two** examples of real-world problems that require numerical methods.

Question 2

State **three** key differences between analytical methods and numerical methods. Present your answer in a table.

Question 3

For each of the following equations, state whether it can be solved analytically or whether a numerical method is required. Justify your answer.

(a) $x^2 - 7x + 10 = 0$
(b) $e^x - 4x = 0$
(c) $3x + 8 = 2x - 1$
(d) $\sin(x) = x^2 - 1$
(e) $x^3 - 2x^2 + x - 1 = 0$

Question 4

Explain the difference between a **direct method** and an **iterative method**. Give one example of each and explain when each type is preferred.

Question 5

Explain in simple terms what it means for a numerical method to be:

(a) **Accurate**
(b) **Stable**
(c) **Convergent**

For each property, give an analogy or real-life example to illustrate your explanation.

9.2 Section B: Analytical Method Practice

Question 6

Solve the following equations analytically (i.e., find the exact solution):

(a) $2x^2 - 8 = 0$
(b) $\frac{1}{x} + \frac{1}{x+1} = \frac{5}{6}$
(c) $4x + 3y = 10$ and $2x - y = 0$ (solve simultaneously)
(d) $\frac{dy}{dx} = 4x^3 - 2x$, with $y(1) = 3$

Question 7

Evaluate the following integrals exactly:

(a) $\int_1^3 (2x + 5)\,dx$

(b) $\int_0^\pi \sin(x)\,dx$

(c) $\int_1^e \frac{1}{x}\,dx$

9.3 Section C: Iterative Method Practice

Question 8

The equation $x^2 - 3 = 0$ can be rearranged as $x = g(x)$ in different ways:

(i) $x = \frac{3}{x}$
(ii) $x = \frac{x^2 + 3}{2x}$ (Newton's method version)
(iii) $x = \sqrt{3}$ (trivial — gives the answer directly but requires a calculator)

For arrangement (ii), perform **5 iterations** of the fixed-point method starting from $x_0 = 2.0$. Record your results in a table with columns: n, x_n, x_{n+1}, $|x_{n+1} - x_n|$.

What do you notice about how quickly the method converges?

Question 9

Use fixed-point iteration with $x_{n+1} = \cos(x_n)$ to solve $\cos(x) = x$.

(a) Start with $x_0 = 0.5$ and perform **8 iterations**. Record results in a table.
(b) Start with $x_0 = 1.5$ and perform **8 iterations**. Record results in a table.
(c) Do both starting values converge to the same answer?
(d) What can you conclude about the role of the initial guess in an iterative method?

Question 10 (*)

Consider the equation $f(x) = x^3 - 2x - 5 = 0$.

(a) Show that there is a root in the interval $[2, 3]$ by evaluating $f(2)$ and $f(3)$.

(b) Rearrange the equation to the form $x = g(x)$ in **two** different ways.

(c) Choose one rearrangement and perform **6 iterations** of fixed-point iteration starting from $x_0 = 2$.

(d) Does your chosen rearrangement converge? Explain what you observe.

9.4 Section D: Discussion and Reflection

Question 11

A student says: *"Since analytical methods give exact answers, they are always better than numerical methods. We should always use them."*

Write a well-reasoned response (4–6 sentences) explaining why you agree or disagree with this statement.

Question 12 (*)

Research (or think carefully about) the following question:

A GPS receiver determines your position by solving a system of equations based on signals from at least 4 satellites. The equations are nonlinear. Which type of method — analytical or numerical, direct or iterative — do you think the GPS device uses? Explain your reasoning.

Question 13 (*)

The **Babylonian method** for computing $\sqrt{N}$ uses the iteration:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

(a) Use this method to compute $\sqrt{7}$, starting from $x_0 = 3$. Perform 5 iterations.
(b) Compare your result with the true value $\sqrt{7} = 2.6457513\ldots$
(c) How many iterations are needed to achieve accuracy to 4 decimal places?
(d) Can you identify which of the *desirable properties* this method possesses?

End of Topic 1 Tutorial Questions

> **Reminder:** Bring your worked solutions to the next seminar session. Be ready to present your solution to Question 9 or Question 13 to the class.