MAT 103: Numerical Analysis I

Topic 5: Numerical Integration

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> *"Integration is easy when you have a formula. Most of the time, you don't."*

1 Introduction

In calculus, you learned to evaluate definite integrals using antiderivatives:

$$\int_a^b f(x)\,dx = F(b) - F(a), \qquad \text{where } F'(x) = f(x)$$

This works beautifully — **when an antiderivative exists and can be found**.

But consider these situations:

- $f(x) = e^{-x^2}$ — there is **no elementary antiderivative** at all. This integral is central to probability and statistics (the normal distribution), yet it cannot be evaluated by the methods of basic calculus.
- $f(x)$ is known only from **experimental measurements** at a few points — there is no formula for $f(x)$, so there is nothing to "integrate" in the usual sense.
- $f(x) = \dfrac{\sin x}{x}$ — has no elementary antiderivative either, despite looking simple.

In all these cases, we need **numerical integration** (also called **numerical quadrature**): a way to compute a good *approximation* to $\int_a^b f(x)\,dx$ using only values of f at a few points.

ℹ The Big Idea

We cannot integrate $f(x)$ directly, but we **can** always integrate a **polynomial**. So the strategy is:

1. Replace $f(x)$ with an interpolating polynomial $P(x)$ that passes through a few known points of f.
2. Integrate $P(x)$ exactly — this is just calculus.
3. Use $\int_a^b P(x)\,dx$ as our approximation to $\int_a^b f(x)\,dx$.

This is exactly the idea you met in Topic 4. Now we put it to work to **derive** the most important integration formulas in numerical analysis: the **Newton–Cotes formulas**, which include the **Trapezoidal Rule** and **Simpson's Rules**.

By the end of this topic, you should be able to:

- Derive the Trapezoidal Rule and Simpson's Rules from interpolating polynomials.
- Apply the Trapezoidal Rule (single and composite) with step-by-step working.
- Apply Simpson's 1/3 Rule and Simpson's 3/8 Rule.
- Estimate and interpret the error in each method.
- Use MATLAB or MAPLE to perform numerical integration.

2 The General Idea — Newton–Cotes Formulas

2.1 Setting Up

Suppose we want to approximate $\int_a^b f(x)\,dx$. We choose $n+1$ equally spaced points:

$$x_0 = a, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad \dots, \quad x_n = b$$

where $h = \dfrac{b-a}{n}$ is the **step size**.

We know the values $f_0 = f(x_0)$, $f_1 = f(x_1)$, ..., $f_n = f(x_n)$.

We replace $f(x)$ with the **Newton Forward interpolating polynomial** $P_n(x)$ through these points (recall Topic 4), and integrate $P_n(x)$ instead of $f(x)$.

2.2 The Substitution Trick

We always use the substitution $x = x_0 + sh$, so that $dx = h\,ds$. When $x = x_0$, $s = 0$. When $x = x_n$, $s = n$.

This turns the integral into:

$$\int_{x_0}^{x_n} f(x)\,dx \;\approx\; \int_{x_0}^{x_n} P_n(x)\,dx = h\int_0^n P_n(s)\,ds$$

Different choices of n (how many points we use) give different formulas — this whole family is called the **Newton–Cotes formulas**:

n	Points used	Resulting formula
1	2 points	**Trapezoidal Rule**
2	3 points	**Simpson's 1/3 Rule**
3	4 points	**Simpson's 3/8 Rule**

We now derive each one **step by step**, starting from the simplest.

3 The Trapezoidal Rule

3.1 Derivation — Step by Step

We use just **2 points**: $x_0 = a$ and $x_1 = b$, with $h = b - a$.

Step 1 — Write the linear (degree 1) interpolating polynomial.

From Topic 4, Newton's Forward formula with $n = 1$ gives:

$$P_1(x) = f_0 + s\,\Delta f_0, \qquad \text{where } \Delta f_0 = f_1 - f_0$$

Step 2 — Substitute $x = x_0 + sh$, so $dx = h\,ds$. The limits become $s = 0$ to $s = 1$.

$$\int_{x_0}^{x_1} f(x)\,dx \approx h\int_0^1 (f_0 + s\,\Delta f_0)\,ds$$

Step 3 — Integrate term by term.

$$h\int_0^1 f_0\,ds = hf_0\Big[s\Big]_0^1 = hf_0$$

$$h\int_0^1 s\,\Delta f_0\,ds = h\,\Delta f_0\left[\frac{s^2}{2}\right]_0^1 = \frac{h}{2}\Delta f_0$$

Step 4 — Add the two pieces and substitute $\Delta f_0 = f_1 - f_0$.

$$\int_{x_0}^{x_1} f(x)\,dx \approx hf_0 + \frac{h}{2}(f_1 - f_0) = \frac{h}{2}(2f_0 + f_1 - f_0) = \frac{h}{2}(f_0 + f_1)$$

ℹ The Trapezoidal Rule

$$\boxed{\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{2}(f_0 + f_1)}$$

where $h = x_1 - x_0$.

Geometric meaning: This is exactly the area of a **trapezoid** with parallel sides f_0 and f_1, and width h — hence the name.

3.2 Example 5.1 — Trapezoidal Rule (Single Panel)

Estimate $\displaystyle\int_0^2 x^2\,dx$ using the Trapezoidal Rule.

(True value: $\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} = 2.666667$)

Step 1 — Identify h, f_0, f_1:

$h = 2 - 0 = 2, \quad f_0 = f(0) = 0^2 = 0, \quad f_1 = f(2) = 2^2 = 4$

Step 2 — Apply the formula:

$$\int_0^2 x^2\,dx \approx \frac{2}{2}(0 + 4) = 1 \times 4 = 4$$

Step 3 — Compare with the true value:

$$\text{Error} = |2.666667 - 4| = 1.333333$$

This is a **large error** — the single trapezoidal panel is a poor approximation here because x^2 curves noticeably over $[0, 2]$.

3.3 Example 5.2 — Trapezoidal Rule (Single Panel)

Estimate $\displaystyle\int_0^1 e^x\,dx$ using the Trapezoidal Rule.

(True value: $e^1 - e^0 = e - 1 = 1.718282$)

$h = 1, \quad f_0 = e^0 = 1, \quad f_1 = e^1 = 2.718282$

$$\int_0^1 e^x\,dx \approx \frac{1}{2}(1+2.718282) = \frac{3.718282}{2} = 1.859141$$

$$\text{Error} = |1.718282 - 1.859141| = 0.140859$$

3.4 Example 5.3 — Trapezoidal Rule (Single Panel)

Estimate $\int_1^2 \frac{1}{x}\,dx$ using the Trapezoidal Rule.

(True value: $\ln(2) - \ln(1) = \ln 2 = 0.693147$)

$h = 1, \quad f_0 = \frac{1}{1} = 1, \quad f_1 = \frac{1}{2} = 0.5$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{2}(1+0.5) = \frac{1.5}{2} = 0.75$$

$$\text{Error} = |0.693147 - 0.75| = 0.056853$$

Why Is the Error So Large?

A single trapezoid uses only **2 points** — it assumes $f(x)$ is a straight line between them. If $f(x)$ curves a lot (like x^2, e^x, or $1/x$), a straight line is a poor fit, and the error is large. The fix: use **several smaller trapezoids** instead of one big one. This is the **Composite Trapezoidal Rule**.

3.5 The Composite Trapezoidal Rule

3.6 Derivation — Step by Step

Instead of one big interval, divide $[a, b]$ into n smaller panels, each of width $h = \frac{b-a}{n}$, with nodes $x_0, x_1, \dots, x_n$.

Step 1 — Apply the Trapezoidal Rule to each small panel separately:

$$\int_{x_0}^{x_1} f\,dx \approx \frac{h}{2}(f_0+f_1), \quad \int_{x_1}^{x_2} f\,dx \approx \frac{h}{2}(f_1+f_2), \quad \dots, \quad \int_{x_{n-1}}^{x_n} f\,dx \approx \frac{h}{2}(f_{n-1}+f_n)$$

Step 2 — Add all the panels together:

$$\int_a^b f\,dx \approx \frac{h}{2}\Big[(f_0+f_1) + (f_1+f_2) + \cdots + (f_{n-1}+f_n)\Big]$$

Step 3 — Notice that every interior f_i (for $i = 1, \dots, n-1$) appears in exactly two consecutive panels, so it is counted twice. Only f_0 and f_n appear once.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{2}\Big[f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\Big]}$$

💡 Easy Way to Remember

Endpoints get weight 1. Every interior point gets weight 2. Then multiply the whole sum by $h/2$.

3.7 Example 5.4 — Composite Trapezoidal Rule

Estimate $\displaystyle\int_0^2 x^2\,dx$ using the Composite Trapezoidal Rule with $n = 4$ panels.

(True value: $8/3 = 2.666667$)

Step 1 — Compute h and the nodes:

$$h = \frac{2-0}{4} = 0.5$$

i	x_i	$f_i = x_i^2$
0	0.0	0.00
1	0.5	0.25
2	1.0	1.00
3	1.5	2.25
4	2.0	4.00

Step 2 — Apply the composite formula (endpoints weight 1, interior weight 2):

$$\int_0^2 x^2\,dx \approx \frac{0.5}{2}\Big[f_0 + 2f_1 + 2f_2 + 2f_3 + f_4\Big]$$

$$= 0.25\Big[0 + 2(0.25) + 2(1.00) + 2(2.25) + 4\Big]$$

$$= 0.25\Big[0 + 0.5 + 2.0 + 4.5 + 4\Big] = 0.25 \times 11 = 2.75$$

Step 3 — Compare with the true value:

$$\text{Error} = |2.666667 - 2.75| = 0.083333$$

Compare with Example 5.1: using **4 panels instead of 1** reduced the error from 1.333333 down to 0.083333 — about **16 times smaller**.

3.8 Example 5.5 — Composite Trapezoidal Rule

Estimate $\displaystyle\int_0^1 e^x\,dx$ using the Composite Trapezoidal Rule with $n = 4$ panels.

(True value: $e - 1 = 1.718282$)

Step 1 — Nodes ($h = 0.25$**):**

i	x_i	$f_i = e^{x_i}$
0	0.00	1.000000
1	0.25	1.284025
2	0.50	1.648721
3	0.75	2.117000
4	1.00	2.718282

Step 2 — Apply the formula:

$$\int_0^1 e^x\,dx \approx \frac{0.25}{2}\Big[1.000000 + 2(1.284025) + 2(1.648721) + 2(2.117000) + 2.718282\Big]$$

$$= 0.125\Big[1.000000 + 2.568050 + 3.297442 + 4.234000 + 2.718282\Big]$$

$$= 0.125 \times 13.817774 = 1.727222$$

Step 3 — Error:

$$\text{Error} = |1.718282 - 1.727222| = 0.008940$$

Compare with Example 5.2 (single panel, error = 0.140859): using 4 panels reduced the error by a factor of about **16**.

3.9 Example 5.6 — Composite Trapezoidal Rule

Estimate $\int_0^1 \frac{1}{1+x^2}\,dx$ using the Composite Trapezoidal Rule with $n = 4$.

(True value: $\arctan(1) - \arctan(0) = \frac{\pi}{4} = 0.785398$)

Step 1 — Nodes ($h = 0.25$):

i	x_i	$f_i = \frac{1}{1+x_i^2}$
0	0.00	1.000000
1	0.25	0.941176
2	0.50	0.800000
3	0.75	0.640000
4	1.00	0.500000

Step 2 — Apply the formula:

$$\approx \frac{0.25}{2}\Big[1.000000 + 2(0.941176) + 2(0.800000) + 2(0.640000) + 0.500000\Big]$$

$$= 0.125\Big[1.000000+1.882352+1.600000+1.280000+0.500000\Big] = 0.125\times 6.262352 = 0.782794$$

Step 3 — Error:

$$\text{Error} = |0.785398 - 0.782794| = 0.002604$$

3.10 Error in the Trapezoidal Rule

ℹ Error Formula (Single Panel)

$$E = -\frac{h^3}{12} f''(\xi), \qquad \xi \in (x_0, x_1)$$

ℹ Error Formula (Composite, n panels)

$$E = -\frac{(b-a)h^2}{12} f''(\xi), \qquad \xi \in (a, b)$$

What this tells us: the error is proportional to h^2. If we **halve** h (i.e. double the number of panels), the error should shrink by a factor of about **4**. Doubling panels from 1 to 4 (halving h twice) should shrink the error by about $4 \times 4 = 16$ — exactly what we observed in Examples 5.1→5.4 and 5.2→5.5!

4 Simpson's 1/3 Rule

4.1 Why a New Formula?

The Trapezoidal Rule fits a **straight line** through 2 points. If $f(x)$ curves, a straight line misses that curvature. **Simpson's Rule** instead fits a **parabola** through 3 points — capturing curvature and giving much better accuracy.

4.2 Derivation — Step by Step

We use **3 points**: $x_0 = a$, $x_1 = x_0 + h$ (midpoint), $x_2 = b = x_0 + 2h$.

Step 1 — Write the quadratic (degree 2) interpolating polynomial (Newton Forward, $n = 2$):

$$P_2(x) = f_0 + s\,\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0$$

Step 2 — Substitute $x = x_0 + sh$, $dx = h\,ds$. Limits: $s = 0$ to $s = 2$.

$$\int_{x_0}^{x_2} f(x)\,dx \approx h \int_0^2 \left[f_0 + s\,\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0\right] ds$$

Step 3 — Integrate term by term.

$$h\int_0^2 f_0\,ds = hf_0[s]_0^2 = 2hf_0$$

$$h\int_0^2 s\,\Delta f_0\,ds = h\,\Delta f_0\left[\frac{s^2}{2}\right]_0^2 = h\,\Delta f_0 \times 2 = 2h\,\Delta f_0$$

For the third term, we need $\int_0^2 s(s-1)\,ds = \int_0^2 (s^2 - s)\,ds = \left[\frac{s^3}{3} - \frac{s^2}{2}\right]_0^2 = \frac{8}{3} - 2 = \frac{2}{3}$

$$h\int_0^2 \frac{s(s-1)}{2}\Delta^2 f_0\,ds = \frac{h\,\Delta^2 f_0}{2} \times \frac{2}{3} = \frac{h\,\Delta^2 f_0}{3}$$

Step 4 — Add the three pieces:

$$\int_{x_0}^{x_2} f(x)\,dx \approx 2hf_0 + 2h\,\Delta f_0 + \frac{h}{3}\Delta^2 f_0$$

Step 5 — Substitute $\Delta f_0 = f_1 - f_0$ and $\Delta^2 f_0 = f_2 - 2f_1 + f_0$, then simplify:

$$= 2hf_0 + 2h(f_1 - f_0) + \frac{h}{3}(f_2 - 2f_1 + f_0)$$

$$= h\left[2f_0 + 2f_1 - 2f_0 + \frac{f_2 - 2f_1 + f_0}{3}\right] = h\left[\frac{f_0}{3} + \frac{4f_1}{3} + \frac{f_2}{3}\right]$$

ℹ Simpson's 1/3 Rule

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2)}$$

where $h = \dfrac{x_2 - x_0}{2}$ and x_1 is the midpoint.

💡 Where Does "1/3" Come From?

The factor $\dfrac{h}{3}$ comes directly from the integral $\int_0^2 \frac{s(s-1)}{2}\,ds = \frac{1}{3}$ that we calculated in Step 3. The pattern of weights $1, 4, 1$ comes from how f_0, f_1, f_2 combine after substituting the differences.

4.3 Example 5.7 — Simpson's 1/3 Rule

Estimate $\displaystyle\int_0^2 x^2\,dx$ using Simpson's 1/3 Rule.

(True value: $8/3 = 2.666667$)

Step 1 — Identify h and the three points:

$h = \dfrac{2-0}{2} = 1$, so $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 4$$

Step 2 — Apply the formula:

$$\int_0^2 x^2\,dx \approx \frac{1}{3}(0+4(1)+4) = \frac{1}{3}(8) = 2.666667$$

Step 3 — Error:

$$\text{Error} = |2.666667 - 2.666667| = 0$$

Simpson's Rule gives the exact answer! This is because $f(x) = x^2$ is degree 2, and Simpson's Rule (built from a degree-2 polynomial) reproduces any quadratic exactly.

4.4 Example 5.8 — Simpson's 1/3 Rule

Estimate $\int_0^1 e^x\,dx$ using Simpson's 1/3 Rule.

(True value: $e - 1 = 1.718282$)

$h = 0.5$, so $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$.

$$f_0 = 1, \quad f_1 = e^{0.5} = 1.648721, \quad f_2 = e^1 = 2.718282$$

$$\int_0^1 e^x\,dx \approx \frac{0.5}{3}\Big(1 + 4(1.648721) + 2.718282\Big)$$

$$= \frac{0.5}{3}\Big(1 + 6.594884 + 2.718282\Big) = \frac{0.5}{3}(10.313166) = 1.718861$$

$$\text{Error} = |1.718282 - 1.718861| = 0.000579$$

Compare with the Trapezoidal Rule single panel (Example 5.2, error = 0.140859): Simpson's Rule with the **same 3 points** is over **240 times more accurate!**

4.5 Example 5.9 — Simpson's 1/3 Rule

Estimate $\int_1^2 \frac{1}{x}\,dx$ using Simpson's 1/3 Rule.

(True value: $\ln 2 = 0.693147$)

$h = 0.5$, so $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$.

$$f_0 = 1, \quad f_1 = \frac{1}{1.5} = 0.666667, \quad f_2 = 0.5$$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.5}{3}\Big(1 + 4(0.666667) + 0.5\Big) = \frac{0.5}{3}(4.166667) = 0.694444$$

$$\text{Error} = |0.693147 - 0.694444| = 0.001297$$

4.6 The Composite Simpson's 1/3 Rule

To get even better accuracy, we apply Simpson's Rule **repeatedly** over pairs of panels. This requires an **even number of panels** n (since each application of Simpson's Rule "uses up" 2 panels at a time).

Pattern of weights: Just like the composite trapezoidal rule had a weight pattern (1, 2, 2, ..., 2, 1), composite Simpson has the pattern:

$$1,\ 4,\ 2,\ 4,\ 2,\ \ldots,\ 4,\ 2,\ 4,\ 1$$

ℹ Composite Simpson's 1/3 Rule

$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{3}\Big[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 4f_{n-1} + f_n\Big]}$$

Odd-indexed points get weight 4. Even-indexed interior points get weight 2. Endpoints get weight 1. Requires n even.

4.7 Example 5.10 — Composite Simpson's 1/3 Rule

Estimate $\displaystyle\int_0^1 e^x\,dx$ using Composite Simpson's 1/3 Rule with $n = 4$.

(True value: $e - 1 = 1.718282$)

Step 1 — Nodes ($h = 0.25$, same table as Example 5.5):

i	x_i	f_i	Weight
0	0.00	1.000000	1
1	0.25	1.284025	4
2	0.50	1.648721	2
3	0.75	2.117000	4
4	1.00	2.718282	1

Step 2 — Apply the formula:

$$\int_0^1 e^x\,dx \approx \frac{0.25}{3}\Big[1.000000+4(1.284025)+2(1.648721)+4(2.117000)+2.718282\Big]$$

$$= \frac{0.25}{3}\Big[1.000000 + 5.136100 + 3.297442 + 8.468000 + 2.718282\Big]$$

$$= \frac{0.25}{3}(20.619824) = 1.718319$$

Step 3 — Error:

$$\text{Error} = |1.718282 - 1.718319| = 0.000037$$

Compare: Trapezoidal with $n = 4$ (Example 5.5) had error 0.008940. Simpson with the **same 4 panels** has error 0.000037 — about **240 times more accurate**, using the exact same data points!

4.8 Example 5.11 — Composite Simpson's 1/3 Rule

Estimate $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$ using Composite Simpson's 1/3 Rule with $n = 4$.

(True value: $\pi/4 = 0.785398$)

Using the same table as Example 5.6:

i	x_i	f_i	Weight
0	0.00	1.000000	1
1	0.25	0.941176	4
2	0.50	0.800000	2
3	0.75	0.640000	4
4	1.00	0.500000	1

$$\int_0^1 \frac{dx}{1+x^2} \approx \frac{0.25}{3}\Big[1.000000+4(0.941176)+2(0.800000)+4(0.640000)+0.500000\Big]$$

$$= \frac{0.25}{3}\Big[1.000000+3.764704+1.600000+2.560000+0.500000\Big] = \frac{0.25}{3}(9.424704) = 0.785392$$

$$\text{Error} = |0.785398 - 0.785392| = 0.000006$$

4.9 Error in Simpson's 1/3 Rule

Error Formula (Single Application)

$$E = -\frac{h^5}{90} f^{(4)}(\xi), \qquad \xi \in (x_0, x_2)$$

Error Formula (Composite, n panels)

$$E = -\frac{(b-a)h^4}{180} f^{(4)}(\xi), \qquad \xi \in (a, b)$$

Key observation: the error is proportional to h^4, not h^2 as in the Trapezoidal Rule. This is why Simpson's Rule is so much more accurate — halving h shrinks the error by a factor of $2^4 = 16$, compared to only $2^2 = 4$ for the Trapezoidal Rule.

4.10 Degree of Precision

A Remarkable Fact

Although Simpson's Rule is derived from a **quadratic** (degree 2) polynomial, it gives the **exact** answer for any polynomial up to **degree 3** — one degree higher than expected! This happens because of a symmetry in the error term that causes the degree-3 contribution to cancel out exactly.

Verification: For $f(x) = x^3$ on $[0, 2]$: true value $= \left[\frac{x^4}{4}\right]_0^2 = 4$. Simpson's Rule with $x_0 = 0, x_1 = 1, x_2 = 2$: $f_0 = 0, f_1 = 1, f_2 = 8$.

$$\frac{1}{3}(0 + 4(1) + 8) = \frac{12}{3} = 4 \quad \checkmark \text{ Exact!}$$

But for $f(x) = x^4$ on $[0, 2]$: true value $= \left[\frac{x^5}{5}\right]_0^2 = 6.4$. Simpson's Rule: $f_0 = 0, f_1 = 1, f_2 = 16$.

$$\frac{1}{3}(0 + 4(1) + 16) = \frac{20}{3} = 6.6\overline{6} \neq 6.4 \quad \text{(not exact — degree 4 fails)}$$

5 Simpson's 3/8 Rule

5.1 Derivation — Step by Step

We use **4 points**: x_0, x_1, x_2, x_3, equally spaced with step $h = \dfrac{x_3 - x_0}{3}$.

Step 1 — Write the cubic (degree 3) interpolating polynomial:

$$P_3(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{6}\Delta^3 f_0$$

Step 2 — Substitute $x = x_0 + sh$, integrate from $s = 0$ to $s = 3$:

$$\int_{x_0}^{x_3} f(x)\,dx \approx h\int_0^3 P_3(s)\,ds$$

Step 3 — Integrate each term (the working is similar to Simpson's 1/3 derivation, but with limits 0 to 3 instead of 0 to 2):

$$\int_0^3 ds = 3, \qquad \int_0^3 s\,ds = \frac{9}{2}, \qquad \int_0^3 \frac{s(s-1)}{2}ds = \frac{9}{4}, \qquad \int_0^3 \frac{s(s-1)(s-2)}{6}ds = \frac{3}{8}$$

Step 4 — Combine and simplify (the algebra is lengthier but follows exactly the same pattern as before — substitute the differences in terms of f_0, f_1, f_2, f_3 and collect like terms):

Simpson's 3/8 Rule

$$\boxed{\int_{x_0}^{x_3} f(x)\,dx \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)}$$

where $h = \dfrac{x_3 - x_0}{3}$.

Where Does "3/8" Come From?

After collecting all the terms in Step 4, the common factor that emerges is $\dfrac{3h}{8}$, with weights $1, 3, 3, 1$ on the four function values. This is why the rule is called "Simpson's 3/8 Rule."

5.2 Example 5.12 — Simpson's 3/8 Rule

Estimate $\displaystyle\int_0^1 e^x\,dx$ using Simpson's 3/8 Rule.

(True value: $e - 1 = 1.718282$)

Step 1 — Identify h and the four points:

$h = \dfrac{1-0}{3} = 0.333333$, so $x_0 = 0,\ x_1 = 0.333333,\ x_2 = 0.666667,\ x_3 = 1.$

$f_0 = 1, \quad f_1 = e^{0.333333} = 1.395612, \quad f_2 = e^{0.666667} = 1.947734, \quad f_3 = e^1 = 2.718282$

Step 2 — Apply the formula:

$$\int_0^1 e^x\,dx \approx \frac{3(0.333333)}{8}\Big(1 + 3(1.395612) + 3(1.947734) + 2.718282\Big)$$

$$= \frac{1}{8}\Big(1 + 4.186836 + 5.843202 + 2.718282\Big) = \frac{1}{8}(13.748320) = 1.718540$$

Step 3 — Error:

$$\text{Error} = |1.718282 - 1.718540| = 0.000258$$

5.3 Example 5.13 — Simpson's 3/8 Rule

Estimate $\displaystyle\int_0^3 x^2\,dx$ using Simpson's 3/8 Rule.

(True value: $\left[\frac{x^3}{3}\right]_0^3 = 9$*)*

$h = 1$, so $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3.$

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 4, \quad f_3 = 9$$

$$\int_0^3 x^2\,dx \approx \frac{3(1)}{8}(0 + 3(1) + 3(4) + 9) = \frac{3}{8}(24) = 9$$

Exact! As expected — Simpson's 3/8 Rule is built from a cubic, so it is exact for any polynomial up to degree 3, and x^2 qualifies.

5.4 Example 5.14 — Simpson's 3/8 Rule

Estimate $\displaystyle\int_1^4 \sqrt{x}\,dx$ using Simpson's 3/8 Rule.

(True value: $\left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = 4.666667$*)*

$h = \dfrac{4-1}{3} = 1$, so $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4.$

$$f_0 = 1, \quad f_1 = \sqrt{2} = 1.414214, \quad f_2 = \sqrt{3} = 1.732051, \quad f_3 = 2$$

$$\int_1^4 \sqrt{x}\, dx \approx \frac{3(1)}{8}\Big(1 + 3(1.414214) + 3(1.732051) + 2\Big)$$

$$= \frac{3}{8}\Big(1 + 4.242642 + 5.196153 + 2\Big) = \frac{3}{8}(12.438795) = 4.664548$$

$$\text{Error} = |4.666667 - 4.664548| = 0.002119$$

5.5 When to Use Simpson's 3/8 Rule

Simpson's 3/8 Rule is mainly useful when the **number of intervals** n is odd (so the basic Simpson's 1/3 Rule, which needs an even n, cannot be applied directly to the whole interval). In that case, we apply Simpson's 3/8 Rule to the **last three intervals**, and Simpson's 1/3 Rule (composite) to the rest.

6 Comparing the Methods

6.1 Summary Table

Rule	Points used	Polynomial degree	Weight pattern	Error (single application)	Exact for degree
Trapezoidal	2	1 (linear)	$1, 1$	$-\frac{h^3}{12} f''(\xi)$	≤ 1
Simpson's 1/3	3	2 (quadratic)	$1, 4, 1$	$-\frac{h^5}{90} f^{(4)}(\xi)$	≤ 3
Simpson's 3/8	4	3 (cubic)	$1, 3, 3, 1$	$-\frac{3h^5}{80} f^{(4)}(\xi)$	≤ 3

6.2 Side-by-Side Comparison

For $\int_0^1 e^x\, dx$ (true value = 1.718282):

Method	Approximation	Absolute Error
Trapezoidal (1 panel)	1.859141	0.140859

Method	Approximation	Absolute Error
Simpson's 1/3 (3 points)	1.718861	0.000579
Simpson's 3/8 (4 points)	1.718540	0.000258
Composite Trapezoidal ($n = 4$)	1.727222	0.008940
Composite Simpson's 1/3 ($n = 4$)	1.718319	0.000037

The Big Picture

- More points → smaller h → smaller error (for any method).
- Higher-degree polynomial → faster error reduction as h shrinks.
- **Simpson's Rule beats the Trapezoidal Rule using the same data points** — it is almost always preferred in practice for smooth functions.

7 Topic Summary

Concept	Formula
General Newton–Cotes idea	Replace $f(x)$ with interpolating polynomial $P_n(x)$, integrate $P_n(x)$ exactly
Trapezoidal Rule	$\frac{h}{2}(f_0 + f_1)$
Composite Trapezoidal	$\frac{h}{2}[f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n]$
Trapezoidal error	$O(h^2)$ — error shrinks by 4× when h halves
Simpson's 1/3 Rule	$\frac{h}{3}(f_0 + 4f_1 + f_2)$
Composite Simpson's 1/3	$\frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + f_n]$ (n even)
Simpson's 1/3 error	$O(h^4)$ — error shrinks by 16× when h halves
Simpson's 3/8 Rule	$\frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)$

Looking Ahead

In **Topic 6**, we move from approximating single integrals to solving entire **systems of linear equations** — using Cramer's Rule, Gaussian Elimination, and LU Decomposition. These methods are foundational to nearly every area of applied mathematics and engineering.

8 Tutorial Questions

Show all working clearly. Questions marked (*) are more challenging.

8.1 Section A: Concepts and Derivation

Question 1

(a) Explain the general idea behind Newton–Cotes integration formulas. Why do we replace $f(x)$ with a polynomial before integrating?

(b) State the substitution used to convert $\int_{x_0}^{x_n} f(x)\,dx$ into an integral with respect to s. Why is this substitution useful?

(c) Without doing the full derivation, explain in your own words why Simpson's Rule (built from a quadratic) is more accurate than the Trapezoidal Rule (built from a line).

Question 2

(a) Starting from the linear interpolating polynomial $P_1(x) = f_0 + s\Delta f_0$, derive the Trapezoidal Rule step by step (show the substitution, the integration, and the simplification).

(b) Starting from the quadratic interpolating polynomial $P_2(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0$, derive Simpson's 1/3 Rule step by step.

8.2 Section B: Trapezoidal Rule

Question 3

Use the Trapezoidal Rule (single panel) to estimate each of the following integrals. For each, compute the exact value and the absolute error.

(a) $\displaystyle\int_0^1 x^3\,dx$

(b) $\displaystyle\int_1^3 \ln(x)\,dx$

(c) $\displaystyle\int_0^{\pi/2} \cos(x)\,dx$

Question 4

Use the Composite Trapezoidal Rule with $n = 4$ to estimate:

$$\int_0^4 \sqrt{x}\,dx$$

(a) Build a table of x_i and f_i values.

(b) Apply the composite formula, showing the weight pattern clearly.

(c) The exact value is $\frac{2}{3}(4)^{3/2} = 5.333333$. Calculate the absolute and percentage error.

Question 5

(a) The Composite Trapezoidal Rule is applied to $\int_0^2 f(x)\,dx$ with $n = 8$, giving an error of 0.04. Using the fact that the error is proportional to h^2, estimate the error if $n = 16$ is used instead (same interval).

(b) If the error must be reduced to below 0.001, approximately how many panels are needed? Show your reasoning.

8.3 Section C: Simpson's 1/3 Rule

Question 6

Use Simpson's 1/3 Rule (single application, 3 points) to estimate each integral. Compute the exact value and the absolute error.

(a) $\int_0^2 x^4\,dx$

(b) $\int_1^3 \frac{1}{x^2}\,dx$

(c) $\int_0^1 \sin(\pi x)\,dx$

Question 7

Use the Composite Simpson's 1/3 Rule with $n = 4$ to estimate:

$$\int_0^1 \frac{1}{1+x}\,dx$$

(a) Build the table of x_i, f_i, and the weight for each point.

(b) Apply the formula and compute the result.

(c) The exact value is $\ln(2) = 0.693147$. Compute the error.

(d) Compare with the Composite Trapezoidal Rule applied to the same data (same $n = 4$ table). Which is more accurate, and by what factor?

Question 8 (*)

(a) Show, by direct calculation, that Simpson's 1/3 Rule gives the **exact** value of $\int_0^2 (3x^3 - x + 1)\,dx$. *(This confirms Simpson's Rule is exact for any cubic.)*

(b) Show that Simpson's 1/3 Rule does **not** give the exact value of $\int_0^2 x^4\,dx$. Compute the actual error.

(c) Based on (a) and (b), state the degree of precision of Simpson's 1/3 Rule.

8.4 Section D: Simpson's 3/8 Rule

Question 9

Use Simpson's 3/8 Rule to estimate:

$$\int_0^3 \frac{1}{1+x}\,dx$$

(a) Identify h and the four nodes x_0, x_1, x_2, x_3.

(b) Apply the formula, showing each step.

(c) The exact value is $\ln(4) = 1.386294$. Compute the absolute error.

Question 10

(a) Use Simpson's 3/8 Rule to estimate $\int_2^5 x^2\,dx$. Verify your answer is exact by comparing with the true value $\left[\frac{x^3}{3}\right]_2^5 = 39$.

(b) Explain, using the degree of precision, why this result must be exact.

8.5 Section E: Comparison and Application

Question 11

The following data table gives values of a function $f(x)$ measured experimentally:

x	0	0.5	1.0	1.5	2.0
$f(x)$	1.0000	1.1180	1.4142	1.8028	2.2361

(No formula is known for $f(x)$ — only this table of measured values.)

(a) Estimate $\int_0^2 f(x)\,dx$ using the Composite Trapezoidal Rule.

(b) Estimate the same integral using the Composite Simpson's 1/3 Rule.

(c) Since there is no formula, we cannot compute the "exact" error. Explain how you would decide which estimate to trust more.

Question 12 (*)

A water tank is being filled, and the rate of inflow $r(t)$ (in litres/minute) is measured every 2 minutes:

t (min)	0	2	4	6	8	10
$r(t)$	12.0	15.5	18.2	16.8	14.1	11.0

(a) Estimate the total volume of water added in the 10-minute period using the Composite Trapezoidal Rule.

(b) Estimate the same quantity using the Composite Simpson's 1/3 Rule. *(Note:* $n = 5$ panels — odd. You will need to handle this by applying Simpson's 1/3 to the first 4 panels and the Trapezoidal Rule to the last panel, OR Simpson's 3/8 to the last 3 panels and Simpson's 1/3 to the first 2.)

(c) Which approach in (b) do you think is more accurate? Explain your reasoning.

Question 13 (*)

A student claims: *"Simpson's Rule is always better than the Trapezoidal Rule, so we should never use the Trapezoidal Rule."*

Write a short response (4–6 sentences) evaluating this claim. Consider: when might the number of available data points force a particular choice? What if $f(x)$ is not smooth (e.g., has a sharp corner)?

End of Topic 5 Tutorial Questions

> **Seminar activity:** Bring your worked solutions for Questions 4, 7, and 11 to the next seminar. Be ready to explain, step by step, how you built your weight pattern for the composite formulas.