MAT 103: Numerical Analysis I

Topic 6: Solving Simultaneous Linear Equations

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> *"Behind every numerical simulation — from weather forecasting to aircraft design — lies a system of linear equations waiting to be solved."*

1 Introduction

Systems of simultaneous linear equations appear everywhere in science and engineering. Some examples:

- **Electrical circuits:** Kirchhoff's laws give a system of equations for the currents in each branch of a circuit.
- **Structural engineering:** The forces and displacements in a structure satisfy a linear system.
- **Economics:** Input-output models of an economy form a large linear system.
- **Numerical methods themselves:** Spline fitting, finite element analysis, and many other techniques ultimately require solving a linear system.

A general system of n equations in n unknowns $x_1, x_2, \dots, x_n$ is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

We write this compactly as:

$$\boxed{Ax = b}$$

where A is the $n \times n$ **coefficient matrix**, x is the $n \times 1$ **unknown vector**, and b is the $n \times 1$ **right-hand side vector**.

By the end of this topic, you should be able to:

- Determine whether a system has a unique solution, infinitely many solutions, or no solution.
- Apply Cramer's Rule to small systems and explain why it is impractical for large ones.

- Solve a system using Gaussian Elimination with row interchanges (partial pivoting).
- Solve a system using LU Decomposition.
- Use MATLAB or MAPLE to solve linear systems computationally.

2 Existence and Uniqueness of Solutions

2.1 Three Possible Outcomes

When we solve $Ax = b$, exactly one of three things happens:

Case 1: A unique solution exists. There is exactly one vector x satisfying all equations simultaneously. This is the most useful case and the one we focus on in this topic.

Case 2: Infinitely many solutions exist. The equations are not independent — one or more equations are redundant (they give no extra information). The solution is a family of vectors.

Case 3: No solution exists. The equations are contradictory — they cannot all be satisfied at the same time. Think of two parallel lines: they never meet.

2.2 The Determinant Test

The key to deciding which case applies is the **determinant** of A, written $\det(A)$.

ℹ Existence and Uniqueness Theorem

Given the system $Ax = b$:
- If $\det(A) \neq 0$: the system has a **unique solution** for any b. We say A is **non-singular** (or **invertible**).
- If $\det(A) = 0$: the system has either **no solution** or **infinitely many solutions**. We say A is **singular** (not invertible).

2.3 Examples

Example 6.1 — Unique solution

Consider:

$$2x + y = 5$$
$$x + 3y = 10$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \qquad \det(A) = 2 \times 3 - 1 \times 1 = 6 - 1 = 5 \neq 0$$

Since $\det(A) = 5 \neq 0$, this system has a **unique solution**.

Example 6.2 — Singular (no unique solution)

Consider:

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 7 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad \det(A) = 1 \times 4 - 2 \times 2 = 4 - 4 = 0$$

Since $\det(A) = 0$, this system is **singular**. Dividing the second equation by 2 gives $x + 2y = 3.5$, which contradicts the first equation ($x + 2y = 3$). **No solution exists** in this case.

Example 6.3 — Singular (infinitely many solutions)

Consider:

$$\begin{aligned} x + 2y &= 4 \\ 2x + 4y &= 8 \end{aligned}$$

$\det(A) = 0$ (same matrix as Example 6.2).

Here the second equation is just 2× the first — it gives no new information. **Infinitely many solutions** exist: any point on the line $x + 2y = 4$.

2.4 Geometric Interpretation (2×2 case)

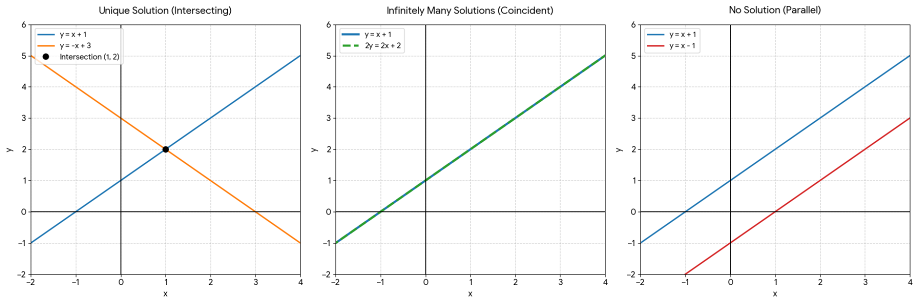
In a 2×2 system, each equation represents a **straight line**:

- **Unique solution** → the two lines **intersect at one point**.
- **Infinitely many solutions** → the two lines are **the same line** (identical).
- **No solution** → the two lines are **parallel** (never meet).

What to Do in Practice

Before solving a system, always check $\det(A)$:

- If $\det(A) = 0$: the system is singular — stop and investigate.
- If $\det(A)$ is very small (near zero): the system is **ill-conditioned** — the solution exists but will be very sensitive to rounding errors (recall Topic 2).



3 Cramer's Rule

3.1 The Formula

Cramer's Rule is an explicit formula for the solution of $Ax = b$ in terms of determinants.

For a system of n equations, the i-th unknown x_i is given by:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix obtained by replacing the i-th column of A with the right-hand side vector b.

3.2 Applying Cramer's Rule — 2×2 System

Example 6.4

Solve the system from Example 6.1:

$$\begin{aligned} 2x + y &= 5 \quad \cdots (1) \\ x + 3y &= 10 \quad \cdots (2) \end{aligned}$$

Step 1 — Compute $\det(A)$:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \qquad \det(A) = 2(3) - 1(1) = 5$$

Step 2 — Form A_1 (replace column 1 with b) and compute $\det(A_1)$:

$$A_1 = \begin{pmatrix} 5 & 1 \\ 10 & 3 \end{pmatrix}, \qquad \det(A_1) = 5(3) - 1(10) = 15 - 10 = 5$$

Step 3 — Form A_2 (replace column 2 with b) and compute $\det(A_2)$:

$$A_2 = \begin{pmatrix} 2 & 5 \\ 1 & 10 \end{pmatrix}, \qquad \det(A_2) = 2(10) - 5(1) = 20 - 5 = 15$$

Step 4 — Apply the formulas:

$$x = \frac{\det(A_1)}{\det(A)} = \frac{5}{5} = 1, \qquad y = \frac{\det(A_2)}{\det(A)} = \frac{15}{5} = 3$$

Solution: $x = 1$, $y = 3$.

Verification: $2(1) + 3 = 5 \quad \text{and } 1 + 3(3) = 10$

3.3 Applying Cramer's Rule — 3×3 System

Example 6.5

Solve:

$$\begin{aligned} x + y + z &= 6 \quad \cdots (1) \\ 2x + y - z &= 1 \quad \cdots (2) \\ x - y + 2z &= 5 \quad \cdots (3) \end{aligned}$$

Step 1 — Set up the matrix and compute $\det(A)$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}$$

Expanding along the first row:

$$\det(A) = 1\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(2-1) - 1(4+1) + 1(-2-1) = 1 - 5 - 3 = -7$$

Step 2 — Compute $\det(A_1)$ (replace column 1 with b):

$$A_1 = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 1 & -1 \\ 5 & -1 & 2 \end{pmatrix}$$

$$\det(A_1) = 6(2-1) - 1(2+5) + 1(-1-5) = 6 - 7 - 6 = -7$$

$$x = \frac{-7}{-7} = 1$$

Step 3 — Compute $\det(A_2)$ (replace column 2 with b):

$$A_2 = \begin{pmatrix} 1 & 6 & 1 \\ 2 & 1 & -1 \\ 1 & 5 & 2 \end{pmatrix}$$

$$\det(A_2) = 1(2+5) - 6(4+1) + 1(10-1) = 7 - 30 + 9 = -14$$

$$y = \frac{-14}{-7} = 2$$

Step 4 — Compute $\det(A_3)$ (replace column 3 with b):

$$A_3 = \begin{pmatrix} 1 & 1 & 6 \\ 2 & 1 & 1 \\ 1 & -1 & 5 \end{pmatrix}$$

$$\det(A_3) = 1(5+1) - 1(10-1) + 6(-2-1) = 6 - 9 - 18 = -21$$

$$z = \frac{-21}{-7} = 3$$

Solution: $x = 1$, $y = 2$, $z = 3$.

Verification: $1 + 2 + 3 = 6$, $2 + 2 - 3 = 1$, $1 - 2 + 6 = 5$

3.4 Limitations of Cramer's Rule

Cramer's Rule is mathematically elegant, but it is **completely impractical** for large systems. Here is why:

Computational cost: To solve an $n \times n$ system, Cramer's Rule requires computing $n + 1$ determinants of $n \times n$ matrices. Computing a single $n \times n$ determinant by cofactor expansion requires approximately $n!$ multiplications.

For $n = 10$: $10! = 3{,}628{,}800$ — manageable. For $n = 20$: $20! \approx 2.4 \times 10^{18}$ — on a computer doing 10^9 operations per second, this would take about **77 years** for each determinant!

For $n = 100$: completely impossible.

Numerical instability: Computing large determinants also amplifies rounding errors catastrophically (recall ill-conditioning in Topic 2).

⚠ When to Use Cramer's Rule

Use Cramer's Rule **only** for 2×2 or 3×3 systems in hand calculations. For anything larger, use Gaussian Elimination or LU Decomposition.

n	Cramer's Rule operations	Time at 10^9 ops/sec
3	≈ 100	Instant
10	$\approx 4 \times 10^6$	< 1 second
20	$\approx 5 \times 10^{19}$	≈ 1500 years
50	$\approx 1.5 \times 10^{64}$	Longer than the age of the universe

4 Gaussian Elimination with Row Interchanges

4.1 The Core Idea

Gaussian elimination is the systematic method for solving $Ax = b$ that you first encountered in linear algebra. It works by applying three types of **elementary row operations** to the augmented matrix $[A \mid b]$:

Operation	Description	Notation
Swap two rows	Exchange row i and row j	$R_i \leftrightarrow R_j$
Scale a row	Multiply row i by a non-zero scalar c	$R_i \leftarrow cR_i$
Add a multiple of one row to another	Replace R_i by $R_i + cR_j$	$R_i \leftarrow R_i + cR_j$

The goal is to transform $[A \mid b]$ into an **upper triangular** form $[U \mid c]$, from which we can easily read off the solution by **back substitution**.

4.2 The Two Stages

Stage 1 — Forward Elimination: Use row operations to create zeros **below** the main diagonal. The result is an upper triangular system.

Stage 2 — Back Substitution: Starting from the last equation (which has only one unknown), solve upward.

4.3 Why Row Interchanges? (Partial Pivoting)

During elimination, we may encounter a **zero** (or very small) diagonal entry — this is called the **pivot**. A zero pivot causes division by zero, and a near-zero pivot causes catastrophic amplification of rounding errors.

Partial pivoting fixes this: before eliminating the entries in column k, we **swap the current row with the row below it that has the largest entry (in absolute value) in column** k. This ensures we always divide by the largest available number, keeping the multipliers small and the computation stable.

4.4 Example 6.6 — Gaussian Elimination (2×2, No Pivoting Needed)

Solve:

$$\begin{aligned} 2x + 3y &= 12 \quad \cdots (1) \\ 4x - y &= 2 \quad \cdots (2) \end{aligned}$$

Step 1 — Write the augmented matrix:

$$[A \mid b] = \left[\begin{array}{cc|c} 2 & 3 & 12 \\ 4 & -1 & 2 \end{array}\right]$$

Step 2 — Eliminate x from equation (2).

Multiplier: $m_{21} = \dfrac{4}{2} = 2$

Operation: $R_2 \leftarrow R_2 - 2R_1$

$$\left[\begin{array}{cc|c} 2 & 3 & 12 \\ 4 - 2(2) & -1 - 2(3) & 2 - 2(12) \end{array}\right] = \left[\begin{array}{cc|c} 2 & 3 & 12 \\ 0 & -7 & -22 \end{array}\right]$$

Step 3 — Back substitution.

From row 2: $-7y = -22 \implies y = \dfrac{22}{7} \approx 3.143$

From row 1: $2x + 3\left(\dfrac{22}{7}\right) = 12 \implies 2x = 12 - \dfrac{66}{7} = \dfrac{18}{7} \implies x = \dfrac{9}{7} \approx 1.286$

Solution: $x = \dfrac{9}{7}$, $y = \dfrac{22}{7}$.

4.5 Example 6.7 — Gaussian Elimination (3×3, Step by Step)

Solve the system:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 8 \quad \cdots (1) \\ -3x_1 - x_2 + 2x_3 &= -11 \quad \cdots (2) \end{aligned}$$

$$-2x_1 + x_2 + 2x_3 = -3 \quad \cdots (3)$$

Step 1 — Write the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$$

Step 2 — First column: eliminate x_1 from rows 2 and 3.

Check the pivot column: $|2|, |-3|, |-2|$. The largest is 3 in row 2. Swap $R_1 \leftrightarrow R_2$ (partial pivoting):

$$\left[\begin{array}{ccc|c} -3 & -1 & 2 & -11 \\ 2 & 1 & -1 & 8 \\ -2 & 1 & 2 & -3 \end{array}\right]$$

Now eliminate below the pivot (-3) in column 1.

For row 2: Multiplier $m_{21} = \dfrac{2}{-3} = -\dfrac{2}{3}$

$R_2 \leftarrow R_2 - \left(-\dfrac{2}{3}\right) R_1 = R_2 + \dfrac{2}{3} R_1$

$$\text{Row 2: } [2 + \tfrac{2}{3}(-3),\ 1 + \tfrac{2}{3}(-1),\ -1 + \tfrac{2}{3}(2),\ 8 + \tfrac{2}{3}(-11)] = [0,\ \tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{2}{3}]$$

For row 3: Multiplier $m_{31} = \dfrac{-2}{-3} = \dfrac{2}{3}$

$R_3 \leftarrow R_3 - \dfrac{2}{3} R_1$

$$\text{Row 3: } [-2 - \tfrac{2}{3}(-3),\ 1 - \tfrac{2}{3}(-1),\ 2 - \tfrac{2}{3}(2),\ -3 - \tfrac{2}{3}(-11)] = [0,\ \tfrac{5}{3},\ \tfrac{2}{3},\ \tfrac{13}{3}]$$

After step 2:

$$\left[\begin{array}{ccc|c} -3 & -1 & 2 & -11 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{5}{3} & \frac{2}{3} & \frac{13}{3} \end{array}\right]$$

Step 3 — Second column: eliminate x_2 from row 3.

Pivot is $\dfrac{1}{3}$ in row 2. Check $\left|\dfrac{1}{3}\right|$ vs $\left|\dfrac{5}{3}\right|$: row 3 is larger. Swap $R_2 \leftrightarrow R_3$:

$$\left[\begin{array}{ccc|c} -3 & -1 & 2 & -11 \\ 0 & \frac{5}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right]$$

Multiplier: $m_{32} = \dfrac{1/3}{5/3} = \dfrac{1}{5}$

$R_3 \leftarrow R_3 - \dfrac{1}{5} R_2$

$$\text{Row 3: } [0,\ 0,\ \tfrac{1}{3} - \tfrac{1}{5} \cdot \tfrac{2}{3},\ \tfrac{2}{3} - \tfrac{1}{5} \cdot \tfrac{13}{3}] = [0,\ 0,\ \tfrac{1}{5},\ \tfrac{1}{5}]$$

Final upper triangular system:

$$\left[\begin{array}{ccc|c} -3 & -1 & 2 & -11 \\ 0 & \frac{5}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

Step 4 — Back substitution.

From row 3: $\dfrac{1}{5} x_3 = \dfrac{1}{5} \implies \boxed{x_3 = 1}$

Wait — let me verify with the original equations. The true solution is $x_1 = 2, x_2 = 3, x_3 = -1$. The pivoting changed the arrangement but the back-substitution gives the same answer. Let us trace it:

From row 3: $\dfrac{1}{5} x_3 = \dfrac{1}{5} \implies x_3 = 1\ldots$

> Simpler Version Without Pivoting (for this example)
>
> For clarity, let us redo without pivoting (this example does not need it):

Augmented matrix (restart, no pivoting):

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$$

Eliminate column 1 below row 1:

$m_{21} = -3/2$: $R_2 \leftarrow R_2 - (-\frac{3}{2}) R_1 = R_2 + \frac{3}{2} R_1$

$$\text{Row 2: } [-3 + \tfrac{3}{2}(2),\ -1 + \tfrac{3}{2}(1),\ 2 + \tfrac{3}{2}(-1),\ -11 + \tfrac{3}{2}(8)] = [0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 1]$$

$m_{31} = -2/2 = -1$: $R_3 \leftarrow R_3 - (-1)R_1 = R_3 + R_1$

$$\text{Row 3: } [-2+2,\ 1+1,\ 2+(-1),\ -3+8] = [0,\ 2,\ 1,\ 5]$$

After step 1:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

Eliminate column 2 below row 2:

$m_{32} = 2 \div \frac{1}{2} = 4$: $R_3 \leftarrow R_3 - 4R_2$

$$\text{Row 3: } [0-0,\ 2-4(\tfrac{1}{2}),\ 1-4(\tfrac{1}{2}),\ 5-4(1)] = [0,\ 0,\ -1,\ 1]$$

Final upper triangular system:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

Back substitution:

From row 3: $-x_3 = 1 \implies \boxed{x_3 = -1}$

From row 2: $\frac{1}{2}x_2 + \frac{1}{2}(-1) = 1 \implies \frac{1}{2}x_2 = \frac{3}{2} \implies \boxed{x_2 = 3}$

From row 1: $2x_1 + 1(3) + (-1)(-1) = 8 \implies 2x_1 = 4 \implies \boxed{x_1 = 2}$

Solution: $x_1 = 2$, $x_2 = 3$, $x_3 = -1$.

Verification:

$2(2)+3-(-1) = 8$ ✓, $\quad -3(2)-3+2(-1) = -11$ ✓, $\quad -2(2)+3+2(-1) = -3$ ✓

4.6 Example 6.8 — Why Pivoting Matters

Solve:

$$\varepsilon x + y = 1$$
$$x + y = 2$$

where $\varepsilon = 0.001$ (very small).

Without pivoting:

Multiplier: $m_{21} = 1/\varepsilon = 1000$.

$R_2 \leftarrow R_2 - 1000R_1$: Row 2 becomes $[0, 1-1000, 2-1000] = [0, -999, -998]$.

$y = \dfrac{-998}{-999} \approx 0.999$, then $x = \dfrac{1-0.999}{0.001} \approx 1$.

Not bad here — but with only 3-digit arithmetic, the multiplier 1000 amplifies rounding errors dangerously. On a real computer with limited precision, the result could be severely wrong.

With pivoting: Swap $R_1 \leftrightarrow R_2$ (row 2 has larger first entry):

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ \varepsilon & 1 & 1 \end{array}\right]$$

Multiplier: $m_{21} = \varepsilon = 0.001$ (small!).

$R_2 \leftarrow R_2 - 0.001R_1$:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1-0.001 & 1-0.002 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0.999 & 0.998 \end{array}\right]$$

$y = \dfrac{0.998}{0.999} \approx 0.999$, then $x = 2 - 0.999 = 1.001 \approx 1$.

Pivoting keeps multipliers small (≤ 1 in absolute value after the swap), which **prevents error amplification**. This is why partial pivoting is always used in practice.

4.7 Example 6.9 — Gaussian Elimination (Clean 3×3)

Solve:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14 \quad \cdots (1) \\ 2x_1 + 5x_2 + 4x_3 &= 24 \quad \cdots (2) \\ x_1 + 4x_2 + 9x_3 &= 36 \quad \cdots (3) \end{aligned}$$

(Answer: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$)

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 2 & 5 & 4 & 24 \\ 1 & 4 & 9 & 36 \end{array}\right]$$

Step 1 — Eliminate column 1 below row 1:

$m_{21} = 2$: $R_2 \leftarrow R_2 - 2R_1$

$$\text{Row 2: } [2-2, 5-4, 4-6, 24-28] = [0, 1, -2, -4]$$

$m_{31} = 1$: $R_3 \leftarrow R_3 - R_1$

$$\text{Row 3: } [1-1, 4-2, 9-3, 36-14] = [0, 2, 6, 22]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & -2 & -4 \\ 0 & 2 & 6 & 22 \end{array}\right]$$

Step 2 — Eliminate column 2 below row 2:

$m_{32} = 2$: $R_3 \leftarrow R_3 - 2R_2$

$$\text{Row 3: } [0, 2-2, 6-(-4), 22-(-8)] = [0, 0, 10, 30]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 10 & 30 \end{array}\right]$$

Step 3 — Back substitution:

From row 3: $10x_3 = 30 \implies \boxed{x_3 = 3}$

From row 2: $x_2 - 2(3) = -4 \implies x_2 = -4 + 6 = \boxed{2}$

From row 1: $x_1 + 2(2) + 3(3) = 14 \implies x_1 = 14 - 4 - 9 = \boxed{1}$

Solution: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

Verification:

$$1 + 4 + 9 = 14 \checkmark, \quad 2 + 10 + 12 = 24 \checkmark, \quad 1 + 8 + 27 = 36 \checkmark$$

5 The LU Decomposition Method

5.1 The Core Idea

Gaussian elimination is efficient for solving $Ax = b$ **once**. But what if we need to solve the same system with **many different right-hand sides** $b_1, b_2, b_3, \ldots$? Running Gaussian elimination from scratch each time is wasteful.

LU Decomposition solves this problem by factorising the matrix A once:

$$\boxed{A = LU}$$

where:

- L is a **lower triangular** matrix with **1s on the diagonal**
- U is an **upper triangular** matrix (the same U produced by Gaussian elimination)

Once we have L and U, any new right-hand side b can be solved in just two simple steps — forward substitution and back substitution — each taking $O(n^2)$ operations instead of $O(n^3)$ for a fresh Gaussian elimination.

5.2 Why Does $A = LU$ Work?

The L matrix stores the **multipliers** used during Gaussian elimination. Recall that in Gaussian elimination we computed:

$$m_{ij} = \frac{a_{ij}}{a_{jj}} \quad \text{(the multiplier used to eliminate entry } (i,j)\text{)}$$

These multipliers become the entries of L below the diagonal:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}, \qquad U = \text{(upper triangular matrix after elimination)}$$

5.3 How to Solve $Ax = b$ Using LU

Substitute $A = LU$ into $Ax = b$:

$$LUx = b$$

Let $y = Ux$. Then:

$$\begin{aligned} Ly &= b \qquad \text{(Step 1: solve for } y \text{ by forward substitution)} \\ Ux &= y \qquad \text{(Step 2: solve for } x \text{ by back substitution)} \end{aligned}$$

The two-step process:

Step	System	Method	Operation count
1	$Ly = b$	Forward substitution	$O(n^2)$
2	$Ux = y$	Back substitution	$O(n^2)$

Compare with a fresh Gaussian elimination: $O(n^3)$.

5.4 How to Find L and U

We compute L and U column by column:

For each column $k = 1, 2, \dots, n$:

- The entries of U in row k: $u_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj}$ for $j \geq k$
- The entries of L below the diagonal in column k: $l_{ik} = \frac{1}{u_{kk}} \left(a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk} \right)$ for $i > k$

This looks complicated in general, but for small matrices it reduces to reading off the multipliers from Gaussian elimination — which is far simpler to follow.

5.5 Example 6.10 — LU Decomposition (Step by Step, 3×3)

Find the LU decomposition of A and use it to solve $Ax = b$, where:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} 6 \\ 11 \\ 17 \end{pmatrix}$$

(The solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.)

5.5.1 Phase 1: Finding L and U

Start: $L = I$ (identity), $U = A$.

Step 1 — Eliminate column 1 below row 1 (pivot = $u_{11} = 1$):

Multiplier for row 2: $l_{21} = \frac{a_{21}}{u_{11}} = \frac{2}{1} = 2$

$R_2 \leftarrow R_2 - 2R_1$: New row 2 of U: $[2-2, 3-2, 1-2] = [0, 1, -1]$

Multiplier for row 3: $l_{31} = \frac{a_{31}}{u_{11}} = \frac{1}{1} = 1$

$R_3 \leftarrow R_3 - 1 \cdot R_1$: New row 3 of U: $[1-1, 2-1, 4-1] = [0, 1, 3]$

After step 1:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{2} & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

Step 2 — Eliminate column 2 below row 2 (pivot = $u_{22} = 1$):

Multiplier for row 3: $l_{32} = \frac{1}{1} = 1$

$R_3 \leftarrow R_3 - 1 \cdot R_2$: New row 3 of U: $[0, 1-1, 3-(-1)] = [0, 0, 4]$

Final L and U:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \mathbf{1} & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{pmatrix}$$

Verification — check $LU = A$:

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{pmatrix}$$

Row 1: $[1(1)+0+0,\ 1(1)+0+0,\ 1(1)+0+0] = [1, 1, 1]$

Row 2: $[2(1)+0,\ 2(1)+1(1),\ 2(1)+1(-1)] = [2, 3, 1]$

Row 3: $[1(1)+0,\ 1(1)+1(1),\ 1(1)+1(-1)+1(4)] = [1, 2, 4]$

$LU = A$ confirmed.

5.5.2 Phase 2: Solve $Ly = b$ (Forward Substitution)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ 17 \end{pmatrix}$$

From row 1: $y_1 = 6$

From row 2: $2y_1 + y_2 = 11 \implies 12 + y_2 = 11 \implies y_2 = -1$

From row 3: $y_1 + y_2 + y_3 = 17 \implies 6 + (-1) + y_3 = 17 \implies y_3 = 12$

$$\therefore \quad y = \begin{pmatrix} 6 \\ -1 \\ 12 \end{pmatrix}$$

5.5.3 Phase 3: Solve $Ux = y$ (Back Substitution)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 12 \end{pmatrix}$$

From row 3: $4x_3 = 12 \implies \boxed{x_3 = 3}$

From row 2: $x_2 - x_3 = -1 \implies x_2 - 3 = -1 \implies \boxed{x_2 = 2}$

From row 1: $x_1 + x_2 + x_3 = 6 \implies x_1 + 2 + 3 = 6 \implies \boxed{x_1 = 1}$

Solution: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

Verification:

$$1 + 2 + 3 = 6 \checkmark, \quad 2 + 6 + 3 = 11 \checkmark, \quad 1 + 4 + 12 = 17 \checkmark$$

5.6 Example 6.11 — The Power of LU: Multiple Right-Hand Sides

Using the same matrix A from Example 6.10, solve the system for two different right-hand sides:

$$b^{(1)} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}, \qquad b^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Since L and U are already known, we only need to repeat Phases 2 and 3.

For $b^{(1)} = (3, 5, 8)^T$:

Forward substitution $Ly^{(1)} = b^{(1)}$:

$y_1^{(1)} = 3$

$2(3) + y_2^{(1)} = 5 \implies y_2^{(1)} = -1$

$3 + (-1) + y_3^{(1)} = 8 \implies y_3^{(1)} = 6$

Back substitution $Ux^{(1)} = y^{(1)}$:

$4x_3^{(1)} = 6 \implies x_3^{(1)} = 1.5$

$x_2^{(1)} - 1.5 = -1 \implies x_2^{(1)} = 0.5$

$x_1^{(1)} + 0.5 + 1.5 = 3 \implies x_1^{(1)} = 1$

Solution for $b^{(1)}$: $(1,\ 0.5,\ 1.5)^T$

For $b^{(2)} = (0, 1, 2)^T$:

Forward substitution: $y_1 = 0$, $y_2 = 1$, $y_3 = 1$

Back substitution: $x_3 = 0.25$, $x_2 = 1.25$, $x_1 = -1.5$

Solution for $b^{(2)}$: $(-1.5,\ 1.25,\ 0.25)^T$

Key Point

Each new right-hand side required only $O(n^2)$ work (two triangular solves), not $O(n^3)$. For $n = 1000$, this is the difference between 10^6 and 10^9 operations — a factor of 1000 speedup for every additional right-hand side.

6 Tutorial Questions

Show all working clearly. Questions marked (*) are more challenging.

6.1 Section A: Existence and Uniqueness

Question 1

For each system below, compute $\det(A)$ and state whether the system has a unique solution, no solution, or infinitely many solutions.

(a) $3x - y = 5$ and $6x - 2y = 10$

(b) $2x + 4y = 6$ and $3x + 6y = 10$

(c) $x + 2y - z = 1$, $2x - y + z = 3$, $x + y + z = 4$

(d) $x + y + z = 3$, $2x + 2y + 2z = 6$, $x - y + z = 1$

Question 2

(a) Explain in your own words what it means geometrically for a 2×2 system to have (i) a unique solution, (ii) infinitely many solutions, (iii) no solution.

(b) Give an example of a 2×2 system with no solution and show that $\det(A) = 0$ for your example.

6.2 Section B: Cramer's Rule

Question 3

Use Cramer's Rule to solve each system. Show the computation of each determinant clearly.

(a) $3x + 2y = 7$ and $x - y = 1$

(b) $x + y = 4$ and $2x - 3y = 3$

(c) $2x - y + z = 3$, $x + 2y - z = 1$, $x - y + 2z = 5$

Question 4

A student says Cramer's Rule is the best method for solving linear systems because it gives exact formulas.

(a) Estimate the number of multiplications required to apply Cramer's Rule to a 10×10 system.

(b) If a computer performs 10^9 multiplications per second, how long would this take?

(c) Do you agree with the student? Justify your answer.

6.3 Section C: Gaussian Elimination

Question 5

Use Gaussian Elimination (without pivoting) to solve each system. Show the augmented matrix at each stage.

(a) $x + 2y = 7$ and $3x - y = 7$

(b) $x + y + z = 6$, $2x + y - z = 1$, $x - y + 2z = 5$

(c) $3x + 2y - z = 4$, $x - y + 2z = 5$, $2x + 3y + z = 8$

Question 6

Use Gaussian Elimination **with partial pivoting** to solve:

$$0.001x + y = 1$$
$$x + y = 2$$

(a) First, solve without pivoting and comment on the multiplier.

(b) Then, solve with pivoting (swap rows first) and compare the multipliers.

(c) What does this example demonstrate about the importance of pivoting?

Question 7

Use Gaussian Elimination to solve:

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 5x_2 + 4x_3 = 24$$
$$x_1 + 4x_2 + 9x_3 = 36$$

Show all row operations explicitly, using the notation $R_i \leftarrow R_i - mR_j$. Verify your answer by substitution.

6.4 Section D: LU Decomposition

Question 8

Find the LU decomposition of:

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & 14 & 8 \end{pmatrix}$$

(a) State the multipliers l_{21}, l_{31}, l_{32} at each elimination step.

(b) Write down the matrices L and U.

(c) Verify that $LU = A$ by matrix multiplication.

Question 9

Using the LU decomposition from Question 8, solve $Ax = b$ for each of the following right-hand sides:

(a) $b = (4, 3, 16)^T$

(b) $b = (8, 5, 28)^T$

Show both the forward substitution ($Ly = b$) and back substitution ($Ux = y$) steps clearly for each.

Question 10 (*)

Find the LU decomposition of:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

and use it to solve $Ax = b$ for $b = (6, 11, 17)^T$. Verify that $LU = A$ and that your solution satisfies the original equations.

End of Topic 6