QMS 101 Introductory Statistics

Topic VI: Probability Theory

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Learning Outcomes

By the end of this topic, you will be able to:

- Explain in plain words what probability means
- Use the basic probability formula and the complement rule
- Tell apart the different types of events (mutually exclusive, independent, etc.)
- Apply the addition rule and the multiplication rule correctly
- Calculate conditional probability and test for independence
- Use Total Probability and Bayes' Theorem
- Calculate expected value and use it to make decisions

Why This Topic Matters

A Question We Face Every Day

You don't know exactly what will happen tomorrow — but you can still **think clearly** about it.

- Will it rain enough this season?
- If one crop fails, how likely is the next one to fail too?
- Should a farmer buy insurance or not?

None of these can be answered with certainty.

But they CAN be answered with **probability** — a number that tells us how likely something is.

A Note Before We Start

You Do Not Need to Be a Mathematician

Probability is something you already think about every day — whether to carry an umbrella, whether to bet on a game, whether to take a risk.

This topic simply gives you a **reliable, structured way** to think about chance.

Every new idea will be explained with:

a story → a simple definition → a worked example → practice

6.1 What is Probability?

Defining Probability

Definition

Probability is simply a number that tells us how likely something is to happen.
It is always between 0 and 1:

$$0 \leq P(A) \leq 1$$

- $P(A) = 0 \rightarrow$ the event is **impossible**
- $P(A) = 1 \rightarrow$ the event is **certain**
- $P(A) = 0.5 \rightarrow$ equally likely to happen or not

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Tip

Easy trick: multiply by 100 to turn any probability into a percentage.
0.30 → 30% chance.

The Probability Scale

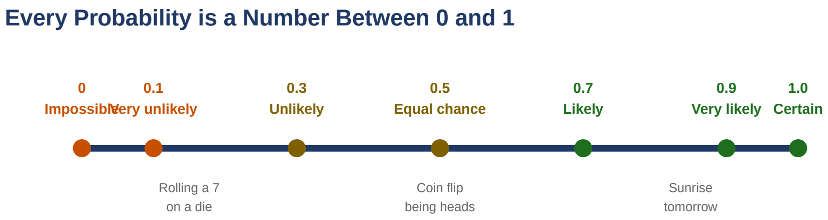

Figure 1

Key Words You Must Know

Four Building-Block Terms

Experiment — any process that produces an outcome.
Example: picking one grain bag and checking it.

Sample space (S) — the list of ALL possible outcomes.
Example: $S = \{\text{damp, dry}\}$

Event (A) — one or more outcomes we care about.
Example: $A = \{\text{damp}\}$

Complement (A') — everything NOT in A.
Example: $A' = \{\text{dry}\}$

The Basic Probability Formula

$$\boxed{P(A) = \frac{\text{Number of outcomes in } A}{\text{Total number of outcomes in } S}}$$

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Example — Grain Bags

A sack has **10 bags**: 3 damp, 7 dry. One bag is picked at random.

$$P(\text{damp}) = \frac{3}{10} = 0.30 \qquad P(\text{dry}) = \frac{7}{10} = 0.70$$

Check: $0.30 + 0.70 = 1.00$

Three Ways to Find Probability

Approach	How it works	Example
Classical	Count favourable ÷ total, when all outcomes are equally likely	Drawing a card, rolling a die
Empirical	Use past data: times it happened ÷ total observations	60 good seasons out of 200 past seasons
Subjective	Expert judgment, no hard data available	"I believe there's a 70% chance of rain"

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Tip

Most farm and business decisions use a mix of **empirical** data and **subjective** judgment.

Probability Axioms

Let S be a sample space and A be an event in S.

The probability $P(A)$ must satisfy three fundamental axioms.

Axiom 1: Non-negativity and Bound

The probability of any event ranges from zero to one.

$$0 \leq P(A) \leq 1$$

Probability Axioms...

Axiom 2: Certainty
The probability of the entire sample space is equal to 1.

$$P(S) = 1$$

Axiom 3: Additivity
If $A_1, A_2, ...$ is a sequence of mutually exclusive events in S, then:

$$P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$$

Properties of Probability with Examples

Examples based on rolling a fair 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$

1. Probability of the Empty Set

An impossible event has a probability of zero.

$$P(\emptyset) = 0 \quad \Longrightarrow \quad \text{Probability of rolling a } 7 = 0$$

2. Complement Rule

The probability of an event *not* occurring.

$$P(A^c) = 1 - P(A) \quad \Longrightarrow \quad P(\text{not rolling a } 6) = 1 - \frac{1}{6} = \frac{5}{6}$$

3. Monotonicity

If event A is nested inside event B, its probability is smaller or equal.

$$\text{If } A \subseteq B \implies P(A) \leq P(B) \quad \implies \quad P(\{2,4\}) \leq P(\{2,4,6\})$$

4. General Addition Law

For overlapping events, subtract the intersection to avoid double-counting.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\text{Example: } P(\text{Even} \cup \text{Greater than 4}) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6}$$

Practice Problems

Apply the properties of probability to solve the following scenarios:

1: A bag contains 5 red marbles and 3 blue marbles. You draw one marble. What is the probability of drawing a green marble?

2: The meteorological department states that there is a 35% chance of rain tomorrow. What is the probability that it will **not** rain tomorrow?

3: Let Event A be drawing the Jack of Spades. Let Event B be drawing any Face Card (Jack, Queen, King). Explain why $P(A) \leq P(B)$ using subsets.

4: In a class of 30 students, 12 play football, 10 play basketball, and 4 play both sports. If a student is chosen at random, what is the probability they play football **or** basketball?

6.2 Types of Events

Mutually Exclusive Events

Definition

Two events are **mutually exclusive** if they **cannot both happen at the same time**.
There is no overlap between them.

Farm Example

A grain bag is either **Good** or **Damaged** — never both.
Good and Damaged are mutually exclusive.

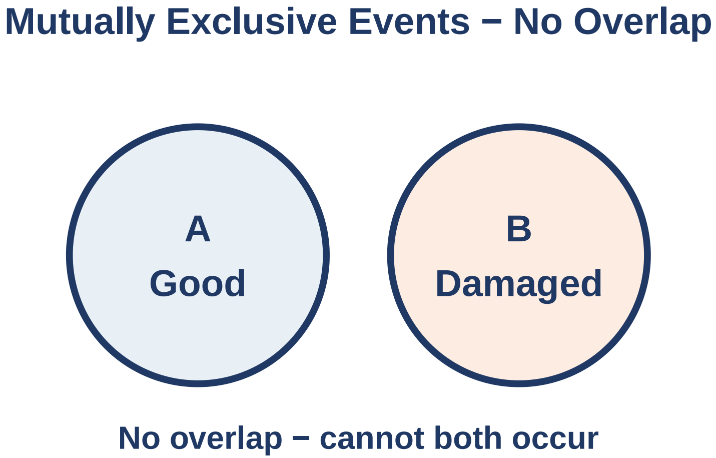
Mutually Exclusive Events – No Overlap
A
Good
B
Damaged
No overlap – cannot both occur

Figure 2

Non-Mutually Exclusive Events

Definition

Two events are **non-mutually exclusive** if they **CAN both happen** at the same time — they overlap.

Farm Example

Among 30 farms: 15 use **irrigation** (A), 18 grow **maize** (B), 8 do **both**.

A farm can irrigate AND grow maize → non-mutually exclusive.

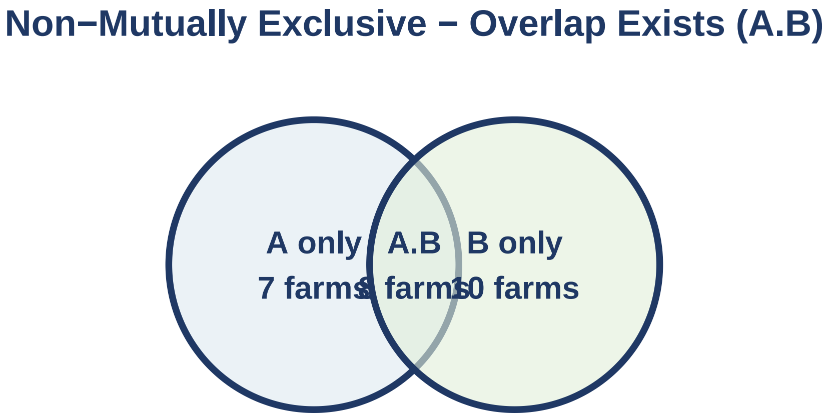
Non−Mutually Exclusive − Overlap Exists (A.B)
A only
7 farms
A.B
8 farms
B only
10 farms

Figure 3

Collectively Exhaustive Events

Definition

A group of events is **collectively exhaustive** if together they cover **every possible outcome** — one of them must happen.

Farm Example

A harvest is **Poor, Average, or Good** — no other outcome exists.

$$P(\text{Poor}) + P(\text{Average}) + P(\text{Good}) = 1$$

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Tip

Mutually exclusive + Collectively exhaustive = a complete, non-overlapping partition.
Their probabilities always sum to exactly 1.

Independent vs. Dependent Events

Independent

Knowing one event happened **does not change** the probability of the other.

Dependent

Knowing one event happened **does change** the probability of the other.

Independent vs. Dependent Events ...

Example of Independent (With Replacement)

- You choose to put the first marble back before picking the second.
- First Draw: You pick a marble. The chance of getting a blue marble is $\frac{3}{10}$. Let's say you pull out a blue one.The Action: You put the blue marble back inside the bag (replacement).
- Second Draw: The bag still has exactly 7 green and 3 blue marbles. The chance of getting a blue marble is still exactly $\frac{3}{10}$.
- Result: The first draw did not change the odds of the second draw.

Example of Dependent (Without Replacement)

- This time, you keep the first marble out.
- First Draw: The bag has 10 marbles. The chance of getting a blue marble is $\frac{3}{10}$. Let's say you pull out a blue one. The Action: You keep the blue marble in your hand and do not put it back.
- Second Draw: The set has changed! The bag now only has 9 marbles total (7 green and 2 blue). The chance of getting a blue marble drops to $\frac{2}{9}$.
- Result: The first draw changed the odds of the second draw.

Quick Summary — Event Types

Type	Meaning	What to do
Mutually exclusive	Cannot happen together	Add probabilities
Non-mutually exclusive	Can happen together	Subtract the overlap
Collectively exhaustive	Covers all outcomes	Probabilities sum to 1
Independent	One doesn't affect the other	Multiply directly
Dependent	One affects the other	Adjust using "given"

Practice 6.2

Your Turn

(a) A student picks one course: Agriculture, Business, or Law. Are "picking Agriculture" and "picking Business" mutually exclusive?

(b) In a group of 40 students: 25 study mornings, 18 exercise, 10 do both.

Are these mutually exclusive? Why or why not?

(c) A bag has 5 red and 3 blue balls. Two are drawn **without replacement**.

(i) $P(\text{red 1st})$ (ii) $P(\text{red 2nd} \mid \text{red 1st})$

(iii) Independent or dependent? Explain.

(d) A harvest is Good (0.45), Average (0.35), or Poor.

(i) Find $P(\text{Poor})$. (ii) Are these mutually exclusive AND collectively exhaustive?

Solved Examples

Question 1: Context & Data

A factory quality check of **100 electronic gadgets** recorded two characteristics: whether the gadget has a **battery defect (B)** and whether it has a **screen scratch (S)**.

	Screen Scratch (S)	No Screen Scratch (S')	Total
Battery Defect (B)	5	15	20
No Battery Defect (B')	10	70	80
Total	15	85	100

Question 1: Tasks

(a) Find the following probabilities:
(i) $P(B)$ — probability of a battery defect.
(ii) $P(S)$ — probability of a screen scratch. (iii) $P(B \cap S)$ — probability of BOTH a battery defect AND a screen scratch.

(b) Are events B and S mutually exclusive? Show your reasoning with numbers.

(c) Use the addition rule to find $P(B \cup S)$ (Battery defect OR screen scratch). Show all steps.

Q1(a): Step-by-Step Solution

- **(i) Find $P(B)$:**
  - Look at the “Total” column for the **Battery Defect (B)** row $\rightarrow$ **20**.
  - Divide by the grand total $\rightarrow$ **100**.
  - $$P(B) = \frac{20}{100} = 0.20$$
- **(ii) Find $P(S)$:**
  - Look at the “Total” row for the **Screen Scratch (S)** column $\rightarrow$ **15**.
  - Divide by the grand total $\rightarrow$ **100**.
  - $$P(S) = \frac{15}{100} = 0.15$$

Q1(a): Step-by-Step Solution (Cont.)

- **(iii) Find $P(B \cap S)$:**
  - Look at the cell where the **Battery Defect (B)** row intersects with the **Screen Scratch (S)** column $\rightarrow$ **5**.
  - Divide by the grand total $\rightarrow$ **100**.
  - $$P(B \cap S) = \frac{5}{100} = 0.05$$

Q1(b): Step-by-Step Solution

Are events B and S mutually exclusive?

- **Definition:** Two events are mutually exclusive if they cannot happen at the same time. This means their intersection must equal zero: $P(B \cap S) = 0$.
- **The data we have:** From part (a)(iii), we know that $P(B \cap S) = 0.05$.
- **Conclusion:**

Since $P(B \cap S) \neq 0$, the events are NOT mutually exclusive.

There are 5 gadgets that have both defects simultaneously.

Q1(c): Step-by-Step Solution

Use the addition rule to find $P(B \cup S)$:

- **Step 1: State the general addition rule formula:**

$$P(B \cup S) = P(B) + P(S) - P(B \cap S)$$

- **Step 2: Substitute your calculated values into the formula:**

$$P(B \cup S) = 0.20 + 0.15 - 0.05 = 0.30$$

Question 2: Context & Data

A school surveyed **80 students** about two characteristics: whether they play **basketball (K)** and whether they swim **competitively (W)**.

	Swims (W)	Does Not Swim (W')	Total
Plays Basketball (K)	0	35	35
No Basketball (K')	15	30	45
Total	15	65	80

Question 2: Tasks

(a) Find the following probabilities:
(i) $P(K)$ — probability a student plays basketball.
(ii) $P(W)$ — probability a student swims competitively.
(iii) $P(K \cap W)$ — probability a student plays basketball AND swims.

(b) Are events K and W mutually exclusive? Show your reasoning with numbers.

(c) Use the addition rule to find $P(K \cup W)$ (Basketball OR Swimming). Show all steps.

Q2(a): Step-by-Step Solution

- **(i) Find $P(K)$:**
  - Look at the "Total" column for the **Plays Basketball (K)** row $\rightarrow$ **35**.
  - Divide by the grand total $\rightarrow$ **80**.
  - $$P(K) = \frac{35}{80} = 0.4375$$
- **(ii) Find $P(W)$:**
  - Look at the "Total" row for the **Swims (W)** column $\rightarrow$ **15**.
  - Divide by the grand total $\rightarrow$ **80**.
  - $$P(W) = \frac{15}{80} = 0.1875$$

Q2(a): Step-by-Step Solution (Cont.)

- **(iii) Find $P(K \cap W)$:**
  - Look at the cell where the **Plays Basketball (K)** row intersects with the **Swims (W)** column → **0**.
  - Divide by the grand total → **80**.
  - $$P(K \cap W) = \frac{0}{80} = 0$$

Q2(b): Step-by-Step Solution

Are events K and W mutually exclusive?

- **Definition:** Two events are mutually exclusive if they cannot happen at the same time $(P(K \cap W) = 0)$.
- **Your Test:** From part (a)(iii), we know that $P(K \cap W) = 0$.
- **Conclusion:**

Since $P(K \cap W) = 0$, the events ARE mutually exclusive.

No students in this survey participate in both sports.

Q2(c): Step-by-Step Solution

Use the addition rule to find $P(K \cup W)$:

- **Step 1: State the general addition rule formula for mutually exclusive events:**

$$P(K \cup W) = P(K) + P(W)$$

- **Step 2: Substitute your calculated fractions into the formula:**

$$P(K \cup W) = \frac{35}{80} + \frac{15}{80} = \frac{50}{80} = 0.625$$

6.6 Mathematical Expectation

The Problem

A Decision Under Uncertainty

A farmer can plant Crop A or Crop B. Profit depends on rainfall, which she cannot control.

Rainfall	Probability	Crop A profit	Crop B profit
Good	0.40	80	50
Average	0.35	50	50
Poor	0.25	10	50

Which crop is better, on average?

The Expected Value Formula

Each outcome is weighted by how likely it is — a weighted average.

$$\boxed{E(X) = \sum x_i \cdot P(x_i)}$$

Plain Meaning

$E(X)$ is the **long-run average** value of X — what you'd expect if the experiment repeated many, many times. It does not have to be a value that can ever actually occur.

Expected Value — Crop A vs. Crop B

Crop A:

x	$P(x)$	$x \cdot P(x)$
80	0.40	32.0
50	0.35	17.5
10	0.25	2.5
		$E(A) = \mathbf{52.0}$

Crop B: always 50, so $E(B) = \mathbf{50.0}$

Decision

Crop A has higher expected profit — but is also riskier (can drop to 10 in a bad year).
Crop B is stable but lower on average. We need a way to measure that risk.

Variance — Measuring Risk

$$\boxed{\text{Var}(X) = E(X^2) - [E(X)]^2} \qquad \text{where } E(X^2) = \sum x_i^2 \cdot P(x_i)$$

Variance for Crop A

x	$P(x)$	x^2	$x^2 P(x)$
80	0.40	6,400	2,560
50	0.35	2,500	875
10	0.25	100	25
			$E(X^2) = 3{,}460$

$$\text{Var}(A) = 3{,}460 - 52^2 = 756 \qquad \text{SD}(A) = \sqrt{756} \approx \mathbf{27.5}$$

For Crop B: always 50 $\rightarrow$ $\text{Var}(B) = 0$, $\text{SD}(B) = 0$.

Comparing Risk and Return

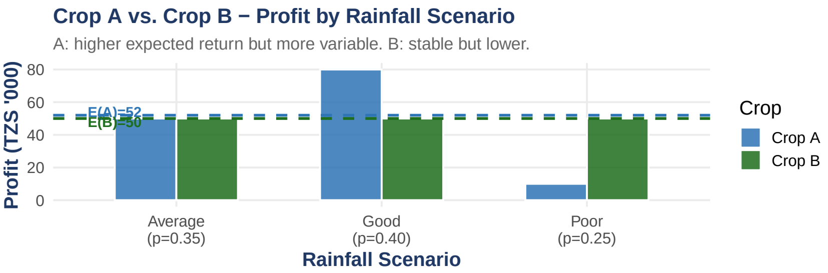

Figure 4

Tip

The key trade-off: higher expected return usually comes with higher risk.
A decision-maker must weigh both — not just the average.

Practice 6.6

Your Turn

A livestock farmer considers insurance. Without insurance:

Event	Probability	Outcome (TZS '000)
No loss	0.70	0
Small loss	0.20	−30
Large loss	0.10	−120

Insurance costs **TZS 15,000/year** and covers all losses.

(a) Calculate the expected loss without insurance.

(b) Calculate the expected outcome with insurance.

(c) Calculate $E(X^2)$, $\text{Var}(X)$, $\text{SD}(X)$ for the uninsured case.

(d) On expected value alone, should she insure? What does SD tell you that E(X) does not?